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The cocenter of the graded affine Hecke algebra and the density theorem

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A B S T R A C T

We determine a basis of the (twisted) cocenter of graded affine Hecke algebras with arbitrary parameters. In this setting, we prove that the kernel of the (twisted) trace map is the commutator subspace (the Density theorem) and that the image is the space of good forms (the trace Paley–Wiener theorem).

1. Introduction

The affine Hecke algebras arise naturally in the theory of smooth representations of reductive p-adic groups. Motivated by the relation to abstract harmonic analysis for p-adic groups (such as the trace Paley–Wiener theorem and the Density theorem [5,17,11,9]), as well as the study of affine Deligne–Lusztig varieties (such as the "dimension = degree" theorem [15, Theorem 6.1]), it is important to describe the cocenter of affine Hecke algebras, i.e., the quotient of the Hecke algebra by the vector subspace spanned by all commutators. In this paper, we solve the related problem for the graded affine Hecke algebras introduced by Lusztig [19].

To describe the results, let $\mathbb{H}$ be the graded Hecke algebra attached to a simple root system Φ and complex parameter function k, Definition 3.1.1. As a $\mathbb{C}$-vector space, $\mathbb{H}$ is isomorphic to $\mathbb{C}[W] \otimes S(V)$, where W is the Weyl group of Φ, and $S(V)$ is the symmetric algebra of V, the underlying (complex) space of the root system.

Let δ be an automorphism of order d of the Dynkin diagram of Φ which preserves the parameters k, and form the extended algebra $\mathbb{H}' = \mathbb{H} \rtimes \langle\delta\rangle$. (When the automorphism δ is trivial, let $\mathbb{H}' = \mathbb{H}$.) The cocenter $\bar{\mathbb{H}}' = \mathbb{H}'/[\mathbb{H}', \mathbb{H}']$ of $\mathbb{H}'$ and the δ-twisted cocenter $\bar{\mathbb{H}}_\delta = \mathbb{H}/[\mathbb{H}, \mathbb{H}]_\delta$ of $\mathbb{H}$ are related in Section 4.1. Recall that

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in Lusztig's reduction theorems [19] from the affine to the graded affine Hecke algebra, nontrivial automorphisms δ appear naturally. This is one motivation for considering the extended graded Hecke algebras $\mathbb{H}'$, rather than just $\mathbb{H}$. In fact, even at the level of the affine Hecke algebra itself, twists by outer automorphisms are unavoidable in relation to various Bernstein components of the category of smooth representations of a reductive p-adic group.

In Section 6.1, we construct a set of elements $\{w_C f_{J_C,i}\}$ of $\mathbb{H}$, where C runs over the δ-twisted conjugacy classes in W. To each class C, we attach a δ-stable subset J_C of the Dynkin diagram, and pick $w_C \in C \cap W_{J_C}$, where W_{J_C} is the parabolic reflection subgroup of W defined by J_C; the elements $f_{J_C,i}$ are chosen in $S(V)$, see Section 6.1 for the precise definitions. Our first result gives a basis for $\bar{\mathbb{H}}_\delta$ (and hence a basis for $\bar{\mathbb{H}}'$), which is independent of the parameter function.

Theorem A. *The set $\{w_C f_{J_C,i}\}$ is a basis for the vector space $\bar{\mathbb{H}}_\delta$.*

The proof that the set $\{w_C f_{J_C,i}\}$ spans $\bar{\mathbb{H}}_\delta$ relies on certain results about δ-twisted conjugacy classes in the Weyl group from Section 2, as well as the use of a filtration in $\mathbb{H}$ and its associated graded object. This allows us to reduce the proof to the case when the parameter function is identically 0. The case $k \equiv 0$ is proved directly in Proposition 6.1.1.

To prove the linear independence, we use the representation theory of $\mathbb{H}$ to produce modules whose traces "separate" the elements $w_C f_{J_C,i}$. This is done in conjunction with a proof of the Density theorem and the (twisted) trace Paley–Wiener theorem for graded Hecke algebras. More precisely, let $R^\delta(\mathbb{H})$ be the $\mathbb{Z}$-span of the δ-stable irreducible $\mathbb{H}$-modules $\mathsf{Irr}^\delta\mathbb{H}$, and let $R^*_\delta(\mathbb{H}) = \mathrm{Hom}_\mathbb{C}(R^\delta(\mathbb{H}), \mathbb{C})$ be the (complex) dual space. The twisted trace map is a linear map

$$\mathrm{tr}^\delta : \bar{\mathbb{H}}_\delta \to R^*_\delta(\mathbb{H}),$$

see Section 5. The map satisfies a certain obvious polynomial condition with respect to the parameters of parabolically induced modules. This requirement leads to the definition of good forms $R^*_\delta(\mathbb{H})_{\mathsf{good}}$ (see Definition 7.1.1), such that the image of tr^δ is automatically in $R^*_\delta(\mathbb{H})_{\mathsf{good}}$.

Theorem B. *The map $\mathrm{tr}^\delta : \bar{\mathbb{H}}_\delta \to R^*_\delta(\mathbb{H})_{\mathsf{good}}$ is a linear isomorphism.*

This is a graded affine Hecke algebra analogue of results from p-adic groups, [5,17,11]. However, our proof (which uses the explicit spanning set of $\bar{\mathbb{H}}_\delta$) is different in several essential places, for example, the proof of injectivity of tr^δ, and the proof of the finiteness of the space of elliptic representations. This approach also leads to the following result on the dimension of the space of δ-elliptic representations $\overline{R}^\delta_0(\mathbb{H})$ (5.2.2).

Theorem C. *The dimension of the δ-twisted elliptic representation space $\overline{R}^\delta_0(\mathbb{H})$ is equal to the number of δ-twisted elliptic conjugacy classes in W.*

When $\delta = 1$ and the parameter function k is positive, this result was previously known from [25], where it was obtained by different methods. Using the explicit description of the cocenter $\bar{\mathbb{H}}_\delta$, we can argue that the dimension is at most the number of δ-elliptic conjugacy classes. To show equality, we construct explicitly in Section 9, via a case-by-case analysis, a set of linearly independent elements of $\overline{R}^\delta_0(\mathbb{H})$ with the desired cardinality and other interesting properties, see Theorem 9.1.1.

Finally, using Clifford theory for $\mathbb{H}'$ and the relation between $\bar{\mathbb{H}}'$, $R(\mathbb{H}')$ and $\bar{\mathbb{H}}_{\delta^i}$, $R^{\delta^i}(\mathbb{H})$ $(i = 1, d)$ respectively, we obtain:

Corollary D. *The trace map $\mathrm{tr} : \bar{\mathbb{H}}' \to R(\mathbb{H}')^*_{\mathsf{good}}$ is a linear isomorphism.*