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Graph-wreath products and finiteness conditions

P.H. Kropholler [*,1], A. Martino

Mathematical Sciences, University of Southampton, Southampton SO17 1BJ, United Kingdom

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ABSTRACT

A notion of *graph-wreath product of groups* is introduced. We obtain sufficient conditions for these products to satisfy the topologically inspired finiteness condition type F_n. Under various additional assumptions we show that these conditions are necessary. Our results generalise results of Bartholdi, Cornulier and Kochloukova about wreath products. Graph-wreath products of groups include classical permutational wreath products and semidirect products of right-angled Artin groups by certain groups of automorphisms amongst others.

1. Introduction

In this paper we introduce a concept of *graph-wreath product of groups* which encompasses the notions of restricted wreath product and of right-angled Artin group as well as other graph products of groups. Our Theorem A gives sufficient conditions for these groups to satisfy the finiteness conditions type F_n, $n \geq 1$. Note that the property F_2 is equivalent to finite presentability, and in the particular case $n = 2$, our result has already been proved for permutational wreath products by Cornulier. Both the results and the methods of the proof in this paper have been influenced by the techniques of [11]. In Theorems B, C, D we examine necessary conditions for the same finiteness properties and here we have built on homological methods of Bartholdi–Cornulier–Kochloukova [1]. The finiteness conditions F_n are discussed in many articles, but we refer to the fundamental paper [4] of Bestvina and Brady for background material because that paper also discusses a number of other ideas relevant to our paper. In [1], results similar to ours but confined to the wreath product case are developed for the properties FP_n rather than F_n. Again, [4] provides an excellent reference for the ideas and remains the fundamental source of examples distinguishing type FP_n and type F_n when $n \geq 2$.

* Corresponding author.
E-mail addresses: p.h.kropholler@soton.ac.uk (P.H. Kropholler), a.martino@soton.ac.uk (A. Martino).
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1.1. Immediate applications

We conclude the paper with some applications to produce new examples. Here is a summary of some of these.

- For arbitrarily large n there exists an elementary amenable group of type F_n which has a subgroup isomorphic to the iterated wreath product

$$\underbrace{(\ldots((\mathbb{Z}\wr\mathbb{Z})\wr\mathbb{Z}))\wr\mathbb{Z}\ldots)\wr\mathbb{Z}}_{n}.$$

  Note that by contrast, every elementary amenable group of type F_∞ is virtually soluble of finite rank and has no sections isomorphic to $A\wr\mathbb{Z}$ for any non-trivial group A. For further information, see [15] where it is shown that finitely generated soluble groups with no large wreath product sections have finite rank, and [17] for an account of elementary amenable groups of type FP_∞.
- There exist groups of type F_∞ which contain infinite normal locally finite subgroups. There exist groups of type F_∞ which contain normal free abelian groups of infinite rank. These examples are again in sharp contrast with results on $\mathbf{H}\mathfrak{F}$-groups of type FP_∞ which state that there is a bound on the orders of finite subgroups and that torsion-free subgroups have finite cohomological dimension. See [16] for an explanation of the class $\mathbf{H}\mathfrak{F}$ and the results that follow for groups in this class.

1.2. Historical remarks

In 1961, Baumslag[2] [2] proved that the standard restricted wreath product $A\wr H$ of two groups A and H is finitely presented if and only if both A and H are finitely presented and either A is trivial or H is finite. One implication here is relatively elementary, namely that $A\wr H$ is finitely presented if A is finitely presented and H is finite. The converse is more subtle and that was the main focus of Baumslag's attention in [2]. Following this initial result, there seems to have been little further progress until, in 1997, the celebrated paper [4] of Bestvina and Brady appeared with the first known examples of group of type FP_2 which are not finitely presented and at this time it therefore became of interest to review many results about finitely presented groups to find out if they were also true of arbitrary groups of type FP_2. In 1998, Baumslag, Bridson and Gruenberg [3] extended Baumslag's earlier result by proving that $A\wr H$ is of type FP_2 if and only if both A and H are of type FP_2 and either $A = 1$ or H is finite.

The subsequent developments appear in the papers already alluded to above, [1,11]. All the results of the four papers [2,3,1,11] concern wreath products. But Cornulier's proof uses an intermediate system of groups between the free group that plays a role in a presentation and the direct sum of groups that plays the role of base of a wreath product. We have developed results which apply to all of these intermediate groups as well as the wreath product.

The graph-wreath product unites the concepts of permutational wreath product of two groups and graph product of a family of groups over a graph. Constructions of this kind have been considered by other authors: see [18] for an example. Our methods build on ideas of Davis, [9].

1.3. Finiteness conditions

We are concerned with the finiteness condition *type* F_n of a group G, meaning that there is an Eilenberg–Mac Lane space with finite n-skeleton. This is a property enjoyed by all finite groups but is topologically

[2] We record with sadness that Gilbert Baumslag died on October 20, 2014, while we were completing the writing of this paper. His work was influential in the field generally and specifically with regard to our project.