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Maximal subalgebras and chief factors of Lie algebras

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ABSTRACT

This paper is a continued investigation of the structure of Lie algebras in relation to their chief factors, using concepts that are analogous to corresponding ones in group theory. The first section investigates the structure of Lie algebras with a core-free maximal subalgebra. The results obtained are then used in section two to consider the relationship of two chief factors of L being L-connected, a weaker equivalence relation on the set of chief factors than that of being isomorphic as L-modules. A strengthened form of the Jordan–Hölder Theorem in which Frattini chief factors correspond is also established for every Lie algebra. The final section introduces the concept of a crown, a notion introduced in group theory by Gaschütz, and shows that it gives much information about the chief factors.

1. Primitive algebras

Throughout L will denote a finite-dimensional Lie algebra over a field F. The symbol '$\oplus$' will denote an algebra direct sum, whilst '$\dot{+}$' will denote a direct sum of the underlying vector space structure alone. If U is a subalgebra of L we define U_L, the *core* (with respect to L) of U to be the largest ideal of L contained in U. We say that U is *core-free* in L if $U_L = 0$. We shall call L *primitive* if it has a core-free maximal subalgebra. The *centraliser* of U in L is $C_L(U) = \{x \in L : [x, U] = 0\}$. Then we have the following characterisation of primitive Lie algebras.

Theorem 1.1.

1. *A Lie algebra L is primitive if and only if there exists a subalgebra M of L such that $L = M + A$ for all minimal ideals A of L.*
2. *Let L be a primitive Lie algebra. Assume that U is a core-free maximal subalgebra of L and that A is a non-trivial ideal of L. Write $C = C_L(A)$. Then $C \cap U = 0$. Moreover, either $C = 0$ or C is a minimal ideal of L.*

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3. *If L is a primitive Lie algebra and U is a core-free maximal subalgebra of L, then exactly one of the following statements holds:*
   (a) *$Soc(L) = A$ is a self-centralising abelian minimal ideal of L which is complemented by U; that is, $L = U \dot{+} A$.*
   (b) *$Soc(L) = A$ is a non-abelian minimal ideal of L which is supplemented by U; that is, $L = U + A$. In this case $C_L(A) = 0$.*
   (c) *$Soc(L) = A \oplus B$, where A and B are the two unique minimal ideals of L and both are complemented by U; that is, $L = A \dot{+} U = B \dot{+} U$. In this case $A = C_L(B)$, $B = C_L(A)$, and A, B and $(A + B) \cap U$ are non-abelian isomorphic Lie algebras.*

Proof.

1. If L is primitive and U is a core-free maximal subalgebra then it is clear that $L = U + A$ for every minimal ideal A of L. Conversely, if there exists a subalgebra M of L such that $L = M + A$ for every minimal ideal A of L and U is a maximal subalgebra of L such that $M \subseteq U$, then U cannot contain any minimal ideal of L, and therefore U is a core-free maximal subalgebra of L.
2. Since U is core-free in L, we have that $L = U + A$. Since A is an ideal of L, then C is an ideal of L and then $C \cap U$ is an ideal of U. Since $[C \cap U, A] = 0$, we have that $C \cap U$ is an ideal of L. Therefore $C \cap U = 0$.
   If $C \neq 0$, consider a minimal ideal X of L such that $X \subseteq C$. Since $X \nsubseteq U$, then $L = X + U$. But now $C = C \cap (X + U) = X + (C \cap U) = X$.
3. Let us assume that A_1, A_2, A_3 are three pairwise distinct minimal ideals of L. Since $A_1 \cap A_2 = A_1 \cap A_3 = A_2 \cap A_3 = 0$, we have that $A_2 \oplus A_3 \subseteq C_L(A_1)$. But then $C_L(A_1)$ is not a minimal ideal of L, and this contradicts 2. Hence, in a primitive Lie algebra there exist at most two distinct minimal ideals.
   Suppose that A is a non-trivial abelian ideal of L. Then $A \subseteq C_L(A)$. Since by 2, $C_L(A)$ is a minimal ideal of L, we have that A is self-centralising. Thus, in a primitive Lie algebra there exists at most one abelian minimal ideal of L. Moreover, $L = A + U$ and A is self-centralising. Then $A \cap U = C_L(A) \cap U = 0$.
   If there exists a unique minimal non-abelian ideal A of L, then $L = A + U$ and $C_L(A) = 0$.
   If there exist two minimal ideals A and B, then $A \cap B = 0$ and then $B \subseteq C_L(A)$ and $A \subseteq C_L(B)$. Since $C_L(A)$ and $C_L(B)$ are minimal ideals of L, we have that $B = C_L(A)$ and $A = C_L(B)$. Now $A \cap U = C_L(B) \cap U = 0$ and $B \cap U = C_L(A) \cap U = 0$. Hence $L = A \dot{+} U = B \dot{+} U$. Since $A = C_L(B)$, it follows that B is non-abelian. Analogously we have that A is non-abelian. Furthermore, we have $A + ((A + B) \cap U) = A + B = B + ((A + B) \cap U)$. Hence

$$A \cong \frac{A}{A \cap B} \cong \frac{A + B}{B} \cong \frac{B + ((A + B) \cap U)}{B} \cong (A + B) \cap U.$$

   Analogously $B \cong (A + B) \cap U$. □

As in the group-theoretic case this leads to three types. A primitive Lie algebra is said to be

1. *primitive of type* 1 if it has a unique minimal ideal that is abelian;
2. *primitive of type* 2 if it has a unique minimal ideal that is non-abelian; and
3. *primitive of type* 3 if it has precisely two distinct minimal ideals each of which is non-abelian.

Of course, primitive Lie algebras of types 2 and 3 are semisimple, and those of types 1 and 2 are monolithic. (A Lie algebra L is called *monolithic* if it has a unique minimal ideal W, the *monolith* of L.) Examples of each type are easy to find.