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The Ekedahl invariants for finite groups

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ABSTRACT

In 2009 Ekedahl introduced certain cohomological invariants of finite groups which are naturally related to the Noether Problem. We show that these invariants are trivial for every finite group in $\mathrm{GL}_3(\mathbf{k})$ and for the fifth discrete Heisenberg group H_5. Moreover in the case of finite linear groups with abelian projective reduction, these invariants satisfy a recurrence relation in a certain Grothendieck group for abelian groups.

Let V be a finite dimension faithful linear representation of a finite group G over a field $\mathbf{k}$ of characteristic prime to the order of G. Inspired by a work of Bergström [1], Ekedahl in [4] and [5] investigated a motivic version of point counting over finite fields. One application of Ekedahl's results is to study when the equality

$$\{\mathrm{GL}(V)/G\} = \{\mathrm{GL}(V)\} \tag{1}$$

holds in the Kontsevich value ring $\widehat{K_0}(\mathbf{Var_k})$ of algebraic $\mathbf{k}$-varieties.

All the known cases where this equality fails are counterexamples to the Noether Problem. In the beginning of the last century, Noether [13] wondered about the rationality of the field extension $\mathbf{k}(V)^G/\mathbf{k}$ for any finite group G and any field $\mathbf{k}$, where $\mathbf{k}(V)^G$ are the invariants of the field of rational functions $\mathbf{k}(V)$ over the regular representation V of G. (The Noether Problem can be stated for any arbitrary field, but we will not need the full generality.)

The first counterexample, $\mathbb{Q}(V)^{\mathbb{Z}/47\mathbb{Z}}/\mathbb{Q}$, was given by Swan in [18] and it appeared during 1969. In the 1980s more counterexamples were found: for every prime p Saltman [16] and Bogomolov [3] showed that there exists a group of order p^9 and, respectively, of order p^6 such that the extension $\mathbb{C}(V)^G/\mathbb{C}$ is not rational.

Saltman used the second unramified cohomology group of the field $\mathbb{C}(V)^G$, $\mathrm{H}^2_{nr}(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z})$, as a cohomological obstruction to rationality. Later, Bogomolov found a group cohomology expression for $\mathrm{H}^2_{nr}(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z})$ which now bears his name and is denoted by $B_0(G)$.

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Recently Hoshi, Kang and Kunyavskii investigated the case where $|G| = p^5$. They showed that $B_0(G) \neq 0$ if and only if G belongs to the isoclinism family ϕ_{10}; see [8].

In 2009, Ekedahl [5] defines, for every integer k, a *cohomological* map

$$\mathcal{H}^k : \widehat{K_0}(\mathbf{Var_k}) \to L_0(\mathbf{Ab}),$$

where $L_0(\mathbf{Ab})$ is the group generated by the isomorphism classes $\{G\}$ of finitely generated abelian groups G under the relation $\{A \oplus B\} = \{A\} + \{B\}$.

Let $\mathbb{L}^i$ be the class of the affine space $\mathbb{A}^i_{\mathbf{k}}$ in $\widehat{K_0}(\mathbf{Var_k})$. In particular, $\mathbb{L}^0$ is the class of a point $\{*\} = \{\mathrm{Spec}(\mathbf{k})\}$. We observe that $\mathbb{L}^i$ is invertible in $\widehat{K_0}(\mathbf{Var_k})$. To define $\mathcal{H}^k$ on $\widehat{K_0}(\mathbf{Var_k})$ is enough to set $\mathcal{H}^k(\{X\}/\mathbb{L}^m) = \{\mathrm{H}^{k+2m}(X;\mathbb{Z})\}$ for every smooth and proper $\mathbf{k}$-variety X (for more details see Section 3 in [11]).

The class $\{\mathcal{B}G\}$ of the classifying stack of G is an element of $\widehat{K_0}(\mathbf{Var_k})$ (see Proposition 2.5.b in [11]) and so one can define:

Definition 1.2. For every integer i, the i-th *Ekedahl invariant* $\mathrm{e}_i(G)$ of the group G is $\mathcal{H}^{-i}(\{\mathcal{B}G\})$ in $L_0(\mathbf{Ab})$. We say that the Ekedahl invariants of G are *trivial* if $\mathrm{e}_i(G) = 0$ for all integers $i \neq 0$.

In Proposition 2.5.a of [11], the author rephrases the equality (1) in terms of algebraic stacks, using the expression

$$\{\mathcal{B}G\} = \frac{\{\mathrm{GL}(V)/G\}}{\{\mathrm{GL}(V)\}} \in \widehat{K_0}(\mathbf{Var_k}).$$

Since $\{\mathrm{GL}(V)\}$ is invertible in $\widehat{K_0}(\mathbf{Var_k})$, the equation (1) holds if and only if $\{\mathcal{B}G\} = \{*\}$ and, if this is the case, then the Ekedahl invariants of G are *trivial*, because $\mathcal{H}^0(\{*\}) = \{\mathbb{Z}\}$ and $\mathcal{H}^k(\{*\}) = 0$ for $k \neq 0$.

These new invariants seem a natural generalization of the Bogomolov multiplier $B_0(G)$ because of the following result.

Theorem 1.3. *(See Thm. 5.1 of [4].) Assume* $\mathrm{char}(\mathbf{k}) = 0$. *If G is a finite group, then $\mathrm{e}_i(G) = 0$ for every $i < 0$, $\mathrm{e}_0(G) = \{\mathbb{Z}\}$, $\mathrm{e}_1(G) = 0$ and $\mathrm{e}_2(G) = \{B_0(G)^\vee\}$, where $B_0(G)^\vee$ is the Pontryagin dual of the Bogomolov multiplier of the group G.*

Moreover, for $i > 0$, the invariant $\mathrm{e}_i(G)$ is an integer linear combination of classes of finite abelian groups.

Using that $\mathrm{e}_2(G) = \{B_0(G)^\vee\}$, one finds some groups with non-trivial Ekedahl invariants (and so $\{\mathcal{B}G\} \neq \{*\}$).

Corollary. *If G is one of the group of order p^9 defined in [16], of order p^6 defined in [3] or of order p^5 belonging to the isoclinism family ϕ_{10} (see [8]), then the second Ekedahl invariant is non-zero and so $\{\mathcal{B}G\} \neq \{*\}$.*

It is not clear if higher Ekedahl invariants are obstructions to the rationality of the extension $\mathbf{k}(V)^G/\mathbf{k}$.

In Corollary 5.8 of [4], it is also proved that $\{\mathcal{B}\mathbb{Z}/47\mathbb{Z}\} \neq \{*\} \in \widehat{K_0}(\mathbf{Var}_{\mathbb{Q}})$, but it is unknown if $\{\mathcal{B}G\} \neq \{*\}$ implies a negative answer to the Noether problem.

To the author's knowledge, there are no examples of finite group G such that $B_0(G) = 0$ (i.e. $\mathrm{e}_2(G) = 0$) and $\mathrm{e}_3(G) \neq 0$. It is worth mentioning that Peyre [14] showed a class of finite groups having $B_0(G) = 0$, but non-trivial third unramified cohomology: it is not clear to the author if this is connected to the non-triviality of higher Ekedahl invariants.