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Chow groups of tensor triangulated categories

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ABSTRACT

We recall P. Balmer's definition of tensor triangular Chow group for a tensor triangulated category $\mathscr{K}$ and explore some of its properties. We give a proof that for a suitably nice scheme X it recovers the usual notion of Chow group from algebraic geometry when we put $\mathscr{K} = \mathrm{D}^{\mathrm{perf}}(X)$. Furthermore, we identify a class of functors for which tensor triangular Chow groups behave functorially and show that (for suitably nice schemes) proper push-forward and flat pull-back of algebraic cycles can be interpreted as being induced by the derived inverse and direct image functors between the bounded derived categories of the involved schemes. We also compute some examples for derived and stable categories from modular representation theory, where we obtain tensor triangular cycle groups with torsion coefficients. This illustrates our point of view that tensor triangular cycles are elements of a certain Grothendieck group, rather than $\mathbb{Z}$-linear combinations of closed subspaces of some topological space.

1. Introduction

A basic topic in algebraic geometry is the study of algebraic cycles on a variety X under the equivalence relation of rational equivalence. This is usually formalized by the Chow group

$$\mathrm{CH}(X) = \bigoplus_p \mathrm{CH}^p(X)$$

where $\mathrm{CH}^p(X)$ is the free abelian group on subvarieties $Y \subset X$ of codimension p, modulo the subgroup of cycles rationally equivalent to zero (i.e. those that appear as the divisor of a rational function on a subvariety of codimension $p - 1$).

How should one approach the subject from the point of view of the derived category of X? In [1], it is shown that we can reconstruct X from its derived category of perfect complexes $\mathrm{D}^{\mathrm{perf}}(X)$ considered as a *tensor triangulated category*. Thus, it should also be possible to reconstruct $\mathrm{CH}(X)$ from $\mathrm{D}^{\mathrm{perf}}(X)$ "in

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purely *tensor triangular* terms". More precisely one would like to construct for each $p \geq 0$ a functor $\mathrm{CH}_p^\Delta(-)$, that takes a tensor triangulated category $\mathscr{K}$ and produces a group $\mathrm{CH}_p^\Delta(\mathscr{K})$ such that $\mathrm{CH}_p^\Delta(\mathrm{D}^{\mathrm{perf}}(X)) \cong \mathrm{CH}^p(X)$.

In this article we show that such a construction is given by a definition of $\mathrm{CH}_p^\Delta(-)$ suggested to the author by P. Balmer in 2011 and now published in [5]. The starting point here is the observation that when one filters the category $\mathrm{D}^{\mathrm{perf}}(X)$ by codimension of support, the successive subquotients split as a coproduct of "local categories" (cf. [2]), analogously to what happens when one performs the same procedure for the abelian category $\mathrm{Coh}(X)$. One continues to define the codimension p cycle group of $\mathrm{D}^{\mathrm{perf}}(X)$ as the Grothendieck group of the p-th subquotient of the filtration. We then obtain a definition of the codimension-p Chow group of $\mathrm{D}^{\mathrm{perf}}(X)$ by analogy with Quillen's coniveau spectral sequence (see [24, §7]). We prove:

Theorem *(3.2.6). Let X be a non-singular, separated scheme of finite type over a field. Endow $\mathrm{D}^{\mathrm{perf}}(X)$ with the opposite of the Krull codimension as a dimension function (cf. Definition 2.2.8). Then for all $p \in \mathbb{Z}$,*

$$\mathrm{CH}_p^\Delta\left(\mathrm{D}^{\mathrm{perf}}(X)\right) \cong \mathrm{CH}^{-p}(X).$$

Apart from reconstructing the classical Chow groups, the definition of $\mathrm{CH}_p^\Delta(\mathscr{K})$ also behaves well in its own right, when we consider it as an invariant of $\mathscr{K}$. We show that $\mathrm{CH}_p^\Delta(-)$ is functorial for the class of exact functors with a relative dimension (see Definition 4.1.1). These are exact (*not necessarily* $\otimes$-exact) functors that preserve the filtration that the choice of a dimension function induces on $\mathscr{T}$, up to a shift by n. We have

Theorem *(4.1.3). Let $F : \mathscr{K} \to \mathscr{L}$ be an exact functor of relative dimension n. Then for all $p \in \mathbb{Z}$, F induces a group homomorphism*

$$\mathrm{CH}_p^\Delta(F) : \mathrm{CH}_p^\Delta(\mathscr{K}) \to \mathrm{CH}_{p+n}^\Delta(\mathscr{L})$$

and we prove that the proper push-forward and flat pull-back morphisms on the classical Chow groups of non-singular varieties can be interpreted as special cases of the above theorem (Theorem 4.3.3 and Theorem 4.4.2).

After investigating a different notion of rational equivalence in Section 5, we then apply our theory outside of algebraic geometry and proceed to compute some examples from modular representation theory: for a finite group G and a field k whose characteristic divides $|G|$, we look at the bounded derived category $\mathrm{D}^{\mathrm{b}}(kG\text{-mod})$ and the stable module category kG-stab. Both categories are naturally tensor triangulated with tensor product $\otimes_k$ and we show that they have isomorphic tensor triangular Chow groups in almost all degrees, which should not come as a big surprise in view of Rickard's equivalence (see [25])

$$kG\text{-stab} \cong \mathrm{D}^{\mathrm{b}}(kG\text{-mod})/\mathrm{D}^{\mathrm{perf}}(kG\text{-mod}) .$$

We prove:

Theorem. *(See Theorem 6.2.7.) Consider kG-stab and $\mathrm{D}^{\mathrm{b}}(kG\text{-mod})$ with the Krull dimension of support as a dimension function on $\mathrm{Spc}(kG\text{-stab})$ and $\mathrm{Spc}(\mathrm{D}^{\mathrm{b}}(kG\text{-mod}))$. Then for all $p \geq 0$, there are isomorphisms*

$$\mathrm{CH}_p^\Delta(kG\text{-stab}) \cong \mathrm{CH}_{p+1}^\Delta(\mathrm{D}^{\mathrm{b}}(kG\text{-mod})) .$$

We then continue to compute the associated tensor triangular Chow groups for $G = \mathbb{Z}/p^n\mathbb{Z}$ and $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$: