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Whittaker modules and quasi-Whittaker modules for the Euclidean Lie algebra $\mathfrak{e}(3)$ ☆

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ABSTRACT

In this paper, we classify simple Whittaker modules and simple quasi-Whittaker modules for the Euclidean Lie algebra $\mathcal{E} = \mathfrak{e}(3)$. We show that a simple $\mathfrak{e}(3)$-module is a Whittaker module if and only if it is a locally finite $\mathcal{E}^+$-module, and it is a quasi-Whittaker module if and only if it is a locally finite $\mathcal{P}$-module. In particular, we get a class of simple weight $\mathfrak{e}(3)$-modules which have infinite-dimensional weight spaces.

1. Introduction

The Euclidean algebra $\mathcal{E} = \mathfrak{e}(3)$ is the complexification of the Lie algebra of the Euclidean group $E(3)$, the Lie group of orientation-preserving isometries of three-dimensional Euclidean space. The representations of $E(3)$ play an important role in the representation theory of the Poincaré group [18]. The Euclidean group has been studied outside of physics and mathematics (see [11,24]). The representation theory of the Euclidean Lie algebra has been studied extensively (see [10,16,25,29]). $\mathcal{E}$ is a special truncated current Lie algebra, whose representations have been studied in [12,13,30,32,34], and have applications in the theory of soliton equations (see [8]) and in the representation theory of affine Kac–Moody Lie algebras. The Euclidean Lie algebra $\mathfrak{e}(3)$ is also a special l-conformal Galilei algebra, the representation theory of which was studied by N. Aizawa, A. Bagchi, R. Gopakumar, J. Lukierski, P.C. Stichel, W.J. Zakrzewski and the others (see [1,3,9,20–22,26], and references therein).

The Whittaker modules for a finite-dimensional complex Lie algebra were introduced by B. Kostant in [17]. Since then results for the complex semisimple Lie algebras have been extended to quantum groups for

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$\mathcal{U}_q(\mathfrak{g})$ [31], and $\mathcal{U}_q(\mathfrak{sl}_2)$ [27], Virasoro algebra [15,19,28], Schrödinger–Witt algebra [36], Heisenberg algebras and affine Kac–Moody algebras [5,14], Heisenberg–Virasoro algebra [7,23], Weyl algebras [4] and some other infinite dimensional Lie algebras [33,35]. Recently, in [6], the authors generalized the concept of Whittaker modules for the Schrödinger algebra. They defined a class of modules called quasi-Whittaker modules. Since the Euclidean Lie algebra and the Schrödinger algebra are l-conformal Galilei algebras with $l = 1$ and $l = \frac{1}{2}$ respectively, we can define quasi-Whittaker modules for the Euclidean Lie algebra as well.

The paper is organized as follows. In Section 2, we give some basic definitions on Whittaker modules and quasi-Whittaker modules. Also we prove some useful lemmas in this section. In Section 3, we classify simple quasi-Whittaker modules. In particular, we give an example of weight $\mathfrak{e}(3)$-modules with an infinite-dimensional weight space. Simple Whittaker modules are classified in Section 4.

In this paper, we denote by $\mathbb{Z}, \mathbb{Z}_+, \mathbb{N}$, $\mathbb{C}$ and $\mathbb{C}^*$ the sets of all integers, nonnegative integers, positive integers, complex numbers, and nonzero complex numbers, respectively. Also, we denote by $\mathbb{C}[x_1, \cdots, x_n]$ the polynomial ring in $x_1, \cdots, x_n$ over $\mathbb{C}$.

2. Preliminaries

The Euclidean Lie algebra $\mathcal{E} = \mathfrak{e}(3)$ is the complex Lie algebra with basis $\{e, h, f, p_+, p, p_-\}$ and the commutation relations:

$$\begin{aligned}
&[h, e] = 2e, && [h, f] = -2f, && [e, f] = h, \\
&[h, p_\pm] = \pm 2p_\pm, && [h, p] = 0, && \\
&[e, p_+] = 0, && [e, p] = -2p_+, && [e, p_-] = p, \\
&[f, p_+] = -p, && [f, p] = 2p_-, && [f, p_-] = 0, \\
&[p_+, p_-] = 0, && [p, p_\pm] = 0. &&
\end{aligned} \tag{2.1}$$

It is easy to see that $\mathcal{E}$ contains two subalgebras: the abelian subalgebra $\mathcal{P} = \mathrm{span}_{\mathbb{C}}\{p, p_-, p_+\}$ and $\mathfrak{sl}_2 = \mathrm{span}_{\mathbb{C}}\{e, h, f\}$. The Euclidean algebra can be viewed as a semidirect product $\mathcal{E} = \mathcal{P} \rtimes \mathfrak{sl}_2$. $\mathcal{E}$ has a triangular decomposition

$$\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^0 \oplus \mathcal{E}^-, \tag{2.2}$$

where $\mathcal{E}^+ = \mathrm{span}_{\mathbb{C}}\{e, p_+\}, \mathcal{E}^0 = \mathrm{span}_{\mathbb{C}}\{h, p\}, \mathcal{E}^- = \mathrm{span}_{\mathbb{C}}\{f, p_-\}$.

The following identities which can be proved by induction will be useful in our proofs.

Lemma 2.1. *For any $k \in \mathbb{Z}_+$, we have*

$$\begin{aligned}
&[e, h^k] = \sum_{i=0}^{k-1} \binom{k}{i} (-2)^{k-i} h^i e, && [e, f^k] = k f^{k-1}(h - k + 1), \\
&[e, p^k] = -2k p^{k-1} p_+, && [e, p_-^k] = k p_-^{k-1} p, \\
&[p_-, h^k] = \sum_{i=0}^{k-1} \binom{k}{i} 2^{k-i} h^i p_-, && [p_-, e^k] = -k e^{k-1} p - k(k-1) e^{k-2} p_+, \\
&[p_+, h^k] = \sum_{i=0}^{k-1} \binom{k}{i} (-2)^{k-i} h^i p_+, && [p_+, f^k] = k f^{k-1} p - k(k-1) f^{k-2} p_-, \\
&[p, e^k] = 2k e^{k-1} p_+, && [p, f^k] = -2k f^{k-1} p_-.
\end{aligned} \tag{2.3}$$