Journal of Pure and Applied Algebra 220 (2016) 1462–1474

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

On algebras of strongly derived unbounded type

Chao Zhang [a,b,*,1]

[a] *Department of Mathematics, Guizhou University, Guiyang 550025, PR China*
[b] *Faculty of Mathematics, Bielefeld University, 33501 Bielefeld, Germany*

ARTICLE INFO

Article history:
Received 1 October 2014
Received in revised form 14 August 2015
Available online 28 September 2015
Communicated by S. Koenig

MSC:
16E05; 16E35; 16G20; 16G60

ABSTRACT

Let A be a finite-dimensional algebra over an algebraically closed field. We prove that A is of strongly derived unbounded type (see Definition 1.1) if and only if there exists an integer m such that $C_m(\text{proj}\,A)$, the category of all minimal projective A-module complexes with degree concentrated in $[0, m]$, is of strongly unbounded type, which is also equivalent to the statement that the repetitive algebra $\hat{A}$ is of strongly unbounded representation type. As a corollary, we can establish the Finite–Strongly unbounded dichotomy on the representation type of $C_m(\text{proj}\,A)$, and also the Discrete–Strongly unbounded dichotomy on the representation type of homotopy category $K^b(\text{proj}\,A)$ and the repetitive algebra $\hat{A}$.

0. Introduction

Throughout this article, k is an algebraically closed field and all the algebras are associative finite-dimensional connected basic k-algebras with identity. In representation theory of algebras, one of the main topics is to study their representation type. As early as 1940s, Brauer and Thrall began the investigation of representation type of finite-dimensional algebras [9,25]. Jans formulated the first and second Brauer–Thrall conjectures in his paper [20]. Roughly speaking, the first Brauer–Thrall conjecture says that an algebra is of bounded representation type if and only if it is of finite representation type, whereas the second Brauer–Thrall conjecture states that the algebras of unbounded representation type are of strongly unbounded representation type. Here, we say an algebra is *of strongly unbounded representation type* if there are infinitely many $d \in \mathbb{N}$ such that for each d, there exist infinitely many isomorphism classes of indecomposable

* Correspondence to: Department of Mathematics, Guizhou University, Guiyang 550025, PR China
E-mail address: zhangc@amss.ac.cn.

[1] The author is sponsored by Sino-Germany (CSC-DAAD) Postdoc scholarship and Research Foundation Project in Guizhou University (2014)01.

http://dx.doi.org/10.1016/j.jpaa.2015.09.012
0022-4049/

modules of dimension d. The study of the Brauer–Thrall conjectures stimulated the development of representation theory of algebras to a large extent [3,4,21,23,24].

In recent years, the bounded derived categories of algebras have been studied widely since Happel's work [17]. By a celebrated theorem from [17], there is a full embedding from the bounded derived category of a finite-dimensional algebra to the stable module category of its repetitive algebra, which is an equivalence if and only if its global dimension is finite. The theorem connects the bounded derived category and the module category, and hence provides a method to explore the property of bounded derived category of algebras by studying their repetitive algebra, for example the derived representation type of algebras [11,15]. Moreover, Vossieck established his definitive work on the classification and distribution of indecomposable objects in the bounded derived category of an algebra in terms of its repetitive algebra. He introduced and classified derived discrete algebras, and proved that an algebra is derived discrete if and only if its repetitive algebra is of discrete representation type [26, Theorem]. Motivated by Vossieck's work, Han and the author introduced the cohomological range of a bounded complex, which leads to the concept of strongly derived unbounded algebras naturally [16, Definition 5]. We say an algebra is *of strongly derived unbounded type* if there are infinitely many $r \in \mathbb{N}$ such that for each r, there exist infinitely many isomorphism classes of indecomposable objects of cohomological range r in its bounded derived category. Han and the author also proved the dichotomy theorem on the representation type of bounded derived category, i.e., a finite-dimensional algebra is either derived discrete or of strongly derived unbounded type, but not both [16, Theorem 2]. The main purpose of this paper is to characterize the strongly derived unbounded algebras using the representation type of their repetitive algebras, which in turn provides a proof of the Discrete–Strongly unbounded dichotomy of the repetitive algebras combined with Han and the author's theorem.

During the research on the bounded derived category $D^b(A)$ of an algebra A, another category turns out to be very crucial, that is $C_m(\operatorname{proj} A)$, the category of all minimal complexes of finite-dimensional projective modules with degree concentrated in $[0, m]$ for any fixed integer $m \geq 0$. Bautista generalized the definition of derived discreteness for the Artin algebras over commutative Artin rings and characterized the derived discrete algebras in terms of generic objects in the category $C_m(\operatorname{proj} A)$ (Ref. [5]). In the context of the representation type, Bautista introduced the finite, tame and wild representation type for $C_m(\operatorname{proj} A)$, and then established the Tame–Wild dichotomy theorem of $C_m(\operatorname{proj} A)$. Moreover, the description that, A is derived discrete if and only if $C_m(\operatorname{proj} A)$ is of finite representation type for all m, is obtained [6]. In present paper, we define the strongly unboundedness of the category $C_m(\operatorname{proj} A)$ for any fixed integer m in a natural way, and describe the algebras of strongly derived unbounded type as those such that the associated category $C_m(\operatorname{proj} A)$ is of strongly unbounded type for some m. The characterization provides us a bridge to connect the strongly unboundedness of bounded derived category $D^b(A)$ and the repetitive algebra $\hat{A}$. Indeed, we prove the following main theorem.

Theorem. *Let A be a finite-dimensional algebra. Then the following statements are equivalent:*

(1) *A is strongly derived unbounded;*
(2) *There exists an integer $m \geq 1$, such that the category $C_m(\operatorname{proj} A)$ is of strongly unbounded type;*
(3) *$K^b(\operatorname{proj} A)$ is of strongly unbounded type;*
(4) *The repetitive algebra $\hat{A}$ is of strongly unbounded representation type.*

By the dichotomy theorem for bounded derived category mentioned above, a finite-dimensional algebra A is derived discrete or of strongly derived unbounded type [16, Theorem 2]. Combined with the equivalent characterizations of derived discrete algebras with representation type of $C_m(\operatorname{proj} A)$ [6], the homotopy category $K^b(\operatorname{proj} A)$ and repetitive algebra $\hat{A}$ [26], we obtain the dichotomy on the representation type of $C_m(\operatorname{proj} A)$, $K^b(\operatorname{proj} A)$ and $\hat{A}$ as follows.