Journal of Pure and Applied Algebra 220 (2016) 1525–1537

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Rings with each right ideal automorphism-invariant

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ARTICLE INFO

Article history:
Received 28 November 2014
Received in revised form 10 August 2015
Available online 26 September 2015
Communicated by V. Suresh

MSC:
16D50; 16U60; 16W20

ABSTRACT

In this paper, we study rings having the property that every right ideal is automorphism-invariant. Such rings are called right a-rings. It is shown that (1) a right a-ring is a direct sum of a square-full semisimple artinian ring and a right square-free ring, (2) a ring R is semisimple artinian if and only if the matrix ring $\mathbb{M}_n(R)$ is a right a-ring for some $n > 1$, (3) every right a-ring is stably-finite, (4) a right a-ring is von Neumann regular if and only if it is semiprime, and (5) a prime right a-ring is simple artinian. We also describe the structure of an indecomposable right artinian right non-singular right a-ring as a triangular matrix ring of certain block matrices.

1. Introduction

The study of rings characterized by homological properties of their one-sided ideals has been an active area of research. Rings for which every right ideal is quasi-injective (known as right q-rings) were introduced by Jain, Mohamed and Singh in [22] and have been studied in a number of other papers (see [3,4,6,16–27, 30,31]) by Beidar, Byrd, Hill, Ivanov and Koehler among other people. In [24] Jain, Singh and Srivastava studied rings whose each right ideal is a finite direct sum of quasi-injective right ideals and called such rings right Σ–q rings. Jain, López-Permouth and Syed in [21] studied rings with each right ideal quasi-continuous and in [7] Clark and Huynh studied rings with each right ideal, a direct sum of quasi-continuous right ideals.

Recall that a module M is called quasi-injective if M is invariant under any endomorphism of its injective envelope; equivalently, any homomorphism from a submodule of M to M extends to an endomorphism of M. As a natural generalization of these modules, Dickson and Fuller [8] initiated the study of modules which are invariant under any automorphism of their injective envelope. These modules have been recently named as automorphism-invariant modules by Lee and Zhou in [29]. In [9] Er, Singh and Srivastava proved that

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http://dx.doi.org/10.1016/j.jpaa.2015.09.016
0022-4049/

a module M is automorphism-invariant if and only if any monomorphism from a submodule of M to M extends to an endomorphism of M thus showing that automorphism-invariant modules are precisely the pseudo-injective modules studied by Jain and Singh in [23] and Teply in [36]. Guil Asensio and Srivastava [13] have shown that automorphism-invariant modules satisfy the full exchange property and these modules also provide a new class of clean modules. The decomposition of automorphism-invariant modules has been described in [9]. If M is an automorphism-invariant module, then M has a decomposition $M = A \oplus B$ where A is quasi-injective and B is square-free. Recall that a module M is called *square-free* if M does not contain a nonzero submodule N isomorphic to $X \oplus X$ for some module X. Recently, Guil Asensio, Keskin Tütüncü and Srivastava [12] have initiated the study of a more general theory of modules invariant under automorphisms of their covers and envelopes. See [1,14,15,33,34] for more details on automorphism-invariant modules.

Rings all of whose right ideals are automorphism-invariant are called *right a-rings* [34]. Since every quasi-injective module is automorphism-invariant, the family of right a-rings includes right q-rings. In fact, the class of right a-rings is a much larger class than the class of right q-rings as there exist examples of rings that are right a-rings but not right q-rings. The goal of this paper is to study these right a-rings and to extend the results in [22] for this new class of rings. In particular, we show that:

(1) A right a-ring is a direct sum of a square-full semisimple artinian ring and a right square-free ring (Theorem 3.4).
(2) A ring R is semisimple artinian if and only if the matrix ring $\mathbb{M}_n(R)$ for some $n > 1$ is an a-ring (Theorem 3.6).
(3) If R is a right a-ring, then R is stably-finite, that is, every matrix ring over R is directly-finite (Theorem 4.3).
(4) A right a-ring is von Neumann regular if and only if it is semiprime (Theorem 4.2), and a prime right a-ring is simple artinian (Theorem 4.7).

We also characterize indecomposable non-local right CS right a-rings. It is shown that:

(5) Let R be an indecomposable, non-local ring. Then R is a right q-ring if and only if R is right CS and a right a-ring (Theorem 4.9).

Let Δ be a right q-ring with an essential maximal right ideal P such that Δ/P is an injective right Δ-module. In a right q-ring, every essential right ideal is two-sided by [22, Theorem 2.3]. Hence Δ/P is a division ring. Let n be an integer with $n \geq 1$, let $D_1, D_2, \ldots, D_n$ be division rings and Δ be a right q-ring, all of whose idempotents are central and the right Δ-module Δ/P is not embeddable into Δ_Δ. Next, let V_i be D_i–D_{i+1}-bimodule such that

$$dim(\{V_i\}_{D_{i+1}}) = 1$$

for all $i = 1, 2, \ldots, n-1$, and let V_n be a D_n–Δ-bimodule such that $V_n P = 0$ and

$$dim(\{V_n\}_{\Delta/P}) = 1.$$

We denote by $G_n(D_1, \ldots, D_n, \Delta, V_1, \ldots, V_n)$, the ring of $(n+1) \times (n+1)$ matrices of the form

$$G_n(D_1, \ldots, D_n, \Delta, V_1, \ldots, V_n) := \begin{pmatrix} D_1 & V_1 & & & & \\ & D_2 & V_2 & & & \\ & & D_3 & V_3 & & \\ & & \cdot & \cdot & \cdot & \\ & & & & \cdot & \\ & & & \cdot & D_n & V_n \\ & & & & & \Delta \end{pmatrix}.$$