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Gotzmann regularity for globally generated coherent sheaves

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ABSTRACT

In this paper, Gotzmann's Regularity Theorem is established for globally generated coherent sheaves on projective space. This is used to extend Gotzmann's explicit construction to the Quot scheme. The Gotzmann representation is applied to bound the second Chern class of a rank 2 globally generated coherent sheaf in terms of the first Chern class.

1. Introduction

Gotzmann gave an explicit construction for the Hilbert scheme $\mathrm{Hilb}_P(\mathbb{P}^n)$ of subschemes of $\mathbb{P}^n$ with Hilbert polynomial P, in [1]. He used a binomial expansion of the Hilbert polynomial to give his Regularity and Persistence theorems as part of this construction; this expansion is called the Gotzmann representation of P. The number of terms in the Gotzmann representation is the Gotzmann number of the Hilbert polynomial. This construction gives $\mathrm{Hilb}_P(\mathbb{P}^n)$ as a subscheme of a cross product of Grassmannians which is determined by the Gotzmann number. Alternatively, it can be given as the degeneracy locus of a single Grassmannian determined by the Gotzmann number.

Some of Gotzmann's results have been extended to other settings; for example, Gasharov [2] proved Gotzmann Persistence for modules. A natural question is: Does Gotzmann Regularity hold for modules? And if so, does Gotzmann's Hilbert scheme construction have an analog for the Quot scheme? Such a construction may have applications to problems in enumerative geometry that reduce to intersections on the Quot scheme, by computing the intersections in the Grassmannian using Schubert calculus; Donaldson–Thomas invariants are a potential example [3].

The author will show that the Hilbert polynomial of an arbitrary module does not have a Gotzmann representation, but a Gotzmann representation exists for the Hilbert polynomial of a globally generated coherent sheaf, and use this to extend Gotzmann's Regularity Theorem to this class of sheaves. This will be used to extend Gotzmann's construction to the Quot scheme $\mathrm{Quot}_P(\mathcal{O}^r_{\mathbb{P}^n})$ of quotients of $\mathcal{O}^r_{\mathbb{P}^n}$ with Hilbert

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polynomial P. An example will show that Gotzmann regularity is not a sharp bound on Castelnuovo–Mumford regularity, as it is in the case of subschemes. In addition, the Gotzmann representation will be applied to bound the second Chern class of a rank 2 globally generated coherent sheaf on $\mathbb{P}^3$ in terms of its first Chern class.

2. Preliminaries

Let k be a field, $S = k[x_0, \dots, x_n]$, and $\mathbb{P}^n = \mathrm{Proj}(S)$. Let $F = Se_1 + \cdots + Se_r$ be the free S-module of rank r with $\deg(e_i) = f_i$ and $f_1 \leq \cdots \leq f_r \leq 0$. A quotient of F corresponds to a coherent sheaf on $\mathbb{P}^n$ that is generated by global sections.

Let N be a graded S-module. The *Hilbert function* of N is $H(N, d) = \dim_k(N_d)$, the dimension of the degree-d part of N as a k-vector space. For $d \gg 0$, the Hilbert function becomes a polynomial $P_N(d)$, the *Hilbert polynomial* of N. For a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$, the Hilbert function and Hilbert polynomial of $\mathcal{F}$ are $H(\mathcal{F}, d) = H(\Gamma^*(\mathcal{F}), d)$ and $P_{\mathcal{F}}(d) = P_{\Gamma^*(\mathcal{F})}(d)$.

Given $a, d \in \mathbb{N}$, the *dth Macaulay representation of a* is the unique expression

$$a = \binom{k(d)}{d} + \binom{k(d-1)}{d-1} + \cdots + \binom{k(\delta)}{\delta},$$

with $\delta \in \mathbb{Z}$, satisfying $k(d) > \cdots > k(\delta) \geq \delta > 0$. Given this representation, the *dth Macaulay transformation of a* is

$$a^{\langle d \rangle} = \binom{k(d)+1}{d+1} + \binom{k(d-1)+1}{d} + \cdots + \binom{k(\delta)+1}{\delta+1}.$$

Example 2.1. The 3rd Macaulay representation of 11 is

$$\binom{5}{3} + \binom{2}{2},$$

and

$$\begin{aligned} 11^{\langle 3 \rangle} &= \binom{6}{4} + \binom{3}{3} \\ &= 16. \end{aligned}$$

Definition 2.2. Given a polynomial $P(d) \in \mathbb{Q}[d]$, a *Gotzmann representation* of P is a binomial expansion

$$P(d) = \binom{d+a_1}{d} + \binom{d+a_2-1}{d-1} + \cdots + \binom{d+a_s-(s-1)}{d-(s-1)},$$

with $a_1, \dots, a_s \in \mathbb{Z}$ and $a_1 \geq \cdots \geq a_s \geq 0$.

Proposition 2.3. *(See [4], Corollary B.31.) The Hilbert polynomial of a subscheme of $\mathbb{P}^n$ has a unique Gotzmann representation.* □

Definition 2.4. The number of terms in the Gotzmann representation of a scheme's Hilbert polynomial is called the *Gotzmann number* of a scheme.

Example 2.5. Assume X is a scheme with Hilbert polynomial $P_X(d) = 3d + 2$. This has Gotzmann representation