Journal of Pure and Applied Algebra 219 (2015) 199–239

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Higher order derived functors and the Adams spectral sequence

Hans-Joachim Baues [a], David Blanc [b,*]

[a] *Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany*
[b] *Department of Mathematics, University of Haifa, 3498838 Haifa, Israel*

A R T I C L E   I N F O

Article history:
Received 10 October 2012
Received in revised form 8 February 2014
Available online 3 May 2014
Communicated by C.A. Weibel

MSC:
Primary: 55T15; secondary: 18G40; 18G50; 55S20

A B S T R A C T

Classical homological algebra studies chain complexes, resolutions, and derived functors in additive categories. In this paper we define *higher order* chain complexes, resolutions, and derived functors in the context of a new type of algebraic structure, called an *algebra of left cubical balls*. We show that higher order resolutions exist in these algebras, and that they determine higher order Ext-groups. In particular, the E_m-term of the Adams spectral sequence ($m > 2$) is such a higher Ext-group, providing a new way of constructing its differentials.

0. Introduction

Topologists have been working on the problem of calculating the homotopy groups of spheres for around eighty years, and many methods have been developed for this purpose. One of the most useful tools for this purpose is the Adams spectral sequence $E_2, E_3, E_4, \ldots$, converging to the p-completed stable homotopy groups of the sphere. Adams computed the E_2-term of the spectral sequence, and showed that it is algebraically determined:

$$E_2^{s,t} \cong \mathrm{Ext}^{s,t}_{\mathcal{A}}(\mathbb{F}_p, \mathbb{F}_p), \tag{0.1}$$

where the derived functor Ext is taken for modules over the mod p Steenrod algebra $\mathcal{A}$ of primary mod p cohomology operations (cf. [1]).

Elements in the mod p cohomology $H^n(X; \mathbb{F}_p)$ of a space X are given by homotopy classes of maps from X to an Eilenberg–Mac Lane space $K(\mathbb{F}_p, n)$, and the elements of the Steenrod algebra are homotopy classes of maps between such Eilenberg–Mac Lane spaces, acting on cohomology classes by composition.

This structure suffices to recover the E_2-term of the Adams spectral sequence, by (0.1). However, in order to determine the higher terms $E_3, E_4, \ldots$, we have to look not only at homotopy classes of maps, but

* Corresponding author.
E-mail addresses: baues@mpim-bonn.mpg.de (H.-J. Baues), blanc@math.haifa.ac.il (D. Blanc).

http://dx.doi.org/10.1016/j.jpaa.2014.04.018
0022-4049/

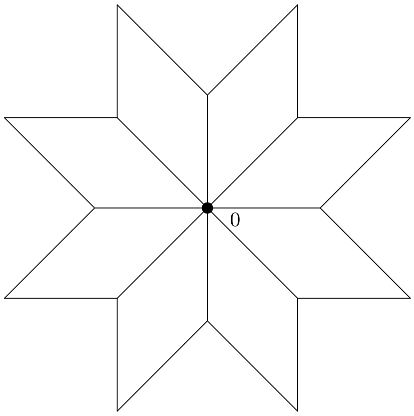

Fig. 1. A left 2-cubical ball.

at the *space* $\mathrm{Map}(X, K(\mathbb{F}_p, n))$ of all maps from X to $K(\mathbb{F}_p, n)$, together with the action of the mapping spaces $\mathrm{Map}(K(\mathbb{F}_p, n), K(\mathbb{F}_p, k))$ on it. This structure is called the *Eilenberg–Mac Lane mapping algebra* of X (cf. [5]).

As we show in this paper, this mapping algebra suffices to compute the whole Adams spectral sequence (cf. [6]). However, this structure is still too complicated for computational purposes, because it involves the *topology* of the mapping spaces, and is not algebraic in nature. Therefore, our main aim here is to extract from these spaces the appropriate algebraic data needed to calculate all the differentials.

For this purpose, we consider singular cubes $\sigma : I^n \to \mathrm{Map}(X, Y)$ in the above mapping spaces. Appropriate collections of such cubes can be glued together to form a *left cubical ball*, as in Fig. 1 (see Section 9 below). Such balls have a new kind of combinatorial-algebraic structure (coming from pasting operations, and so on), which we call an *algebra of left n-cubical balls*. The case $n = 1$ is discussed in detail in Section 12, where we show that an algebra of left 1-cubical balls corresponds to the notion of an abelian track category.

In such algebras, we define the notion of a higher order chain complex, replacing the equation $\partial_{n-1} \circ \partial_n = 0$ in an ordinary chain complex by nullhomotopies $H_n : \partial_{n-1} \circ \partial_n \sim 0$, and higher nullhomotopies (see Section 3). This allows us to define higher order resolutions, in Section 8, and show:

Theorem A. *Higher order resolutions exist in any algebra of left n-cubical balls.*

See Theorem 14.5 below.

It turns out that such higher order resolutions provide a method for calculating the higher terms of the Adams spectral sequence, which thus may be identified as certain m-th order derived functors, called m-th order Ext-groups:

Theorem B. *Any n-th order resolution determines the higher order* Ext*-groups E_m for $m \le n + 2$. In the algebra of left cubical balls determined by the Eilenberg–Mac Lane mapping algebra, these higher order* Ext*-groups compute the E_m-terms of the Adams spectral sequence for $m \le n + 2$.*

See Theorems 15.9 and 15.11 below.

0.2. Remark. As shown in [10], at the secondary level (that is, computing the d_2-differential), this algebra of cubical balls can be replaced by an ordinary differential graded algebra. In [8], the first author and Martin Frankland show that this can also be done at the tertiary level. It is conjectured in [4] that there is a single differential graded algebra, extracted from the algebra of cubical balls, from which the whole Adams spectral sequence can be computed.