Journal of Pure and Applied Algebra 219 (2015) 331–397

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

An algebraic geometric model of an action of the face monoid associated to a Kac–Moody group on its building

C. Mokler

Fakultät für Mathematik, Universität Bochum, Universitätsstraße 150, 44780 Bochum, Germany

ARTICLE INFO

Article history:
Received 1 May 2013
Received in revised form 24 March 2014
Available online 20 May 2014
Communicated by S. Donkin

MSC:
20G44; 14M15; 51E24; 20M32

ABSTRACT

The face monoid $\widehat{G}$ described in [11] acts on the integrable irreducible highest weight modules of a symmetrizable Kac–Moody algebra. It has similar structural properties as a reductive algebraic monoid whose unit group is a symmetrizable Kac–Moody group G. We found in [15] two natural extensions of the action of the Kac–Moody group G on its building Ω to actions of the face monoid $\widehat{G}$ on the building Ω. Now we give an algebraic geometric model of one of these actions of the face monoid $\widehat{G}$ on Ω, where the building Ω is obtained as a part of the $\mathbb{F}$-valued points of the spectrum of all homogeneous prime ideals of the Cartan algebra CA of the Kac–Moody group G. We describe the spectrum of all homogeneous prime ideals of the Cartan algebra CA and determine its $\mathbb{F}$-valued points.

0. Introduction

The face monoid $\widehat{G}$ is an infinite-dimensional algebraic monoid. It has been obtained in [11] by a Tannaka reconstruction from categories determined by the integrable irreducible highest weight representations of a symmetrizable Kac–Moody algebra. By its construction and by the involved categories it is a very natural object. Its Zariski open dense unit group G coincides, up to a slightly extended maximal torus, with the Kac–Moody group defined representation theoretically in [4].

The face monoid is a purely infinite-dimensional phenomenon, quite unexpected. In the classical case, i.e., if one takes a split semisimple Lie algebra for the symmetrizable Kac–Moody algebra, it coincides with a split semisimple simply connected algebraic group. Put in another way, there seem to exist fundamentally different infinite-dimensional generalizations of a split semisimple simply connected algebraic group.

The results obtained in [11–14] show that the face monoid $\widehat{G}$ has similar structural and algebraic geometric properties as a reductive algebraic monoid, e.g., the monoid of $(n\times n)$-matrices. The face monoid $\widehat{G}$ is the first example of an infinite-dimensional reductive algebraic monoid. Actually, it is particular. The investigation

E-mail address: claus.mokler@web.de.

http://dx.doi.org/10.1016/j.jpaa.2014.04.024
0022-4049/

of the conjugacy classes in [16] will show that the relation between the face monoid $\widehat{G}$ and its unit group G, the Kac–Moody group, is much closer than for a general reductive algebraic monoid.

Obviously, there is the following question: Does the face monoid $\widehat{G}$ fit in some way to the building theory of the Kac–Moody group G? In [15] we investigated how to extend the natural action of the Kac–Moody group G on its building Ω to actions of the face monoid $\widehat{G}$ on Ω. To explain the results obtained in [15] note the following facts:

For the face monoid $\widehat{G}$ an infinite Renner monoid $\widehat{\mathcal{W}}$ plays the same role as the Weyl group $\mathcal{W}$ does for the Kac–Moody group G. For example, there are Bruhat and Birkhoff decompositions of $\widehat{G}$, similar as for G, but the Weyl group $\mathcal{W}$ is replaced by the monoid $\widehat{\mathcal{W}}$. The monoid $\widehat{\mathcal{W}}$ can be constructed from the Weyl group $\mathcal{W}$ and the face lattice of the Tits cone X, where the term "face" means a face of the convex cone X in the sense of convex geometry. The Weyl group $\mathcal{W}$ is the unit group of $\widehat{\mathcal{W}}$.

The building Ω is covered by certain subcomplexes, the apartments, which are isomorphic to the Coxeter complex $\mathcal{C}$ associated to the Weyl group $\mathcal{W}$. This connects the action of G on Ω and the action of $\mathcal{W}$ on $\mathcal{C}$.

Now let $\mathcal{A}$ be the standard apartment associated to a fixed BN-pair (B, N) of the Kac–Moody group G. The group N may be obtained as normalizer $N = N_G(T)$ of the standard torus $T = B \cap N$, and the Weyl group $\mathcal{W}$ identifies with N/T. If we define similarly $\widehat{N} := N_{\widehat{G}}(T)$, then the monoid $\widehat{\mathcal{W}}$ identifies with $\widehat{N}/T$. Consider the following diagram:

$$\begin{array}{ccccc} \Omega & \supset & \mathcal{A} & \cong & \mathcal{C} \\ \circlearrowleft & & \circlearrowleft & & \circlearrowleft \\ G & \supset & N & \twoheadrightarrow & \mathcal{W} \\ \cap & & \cap & & \cap \\ \widehat{G}\,? & \supset & \widehat{N} & \twoheadrightarrow & \widehat{\mathcal{W}} \end{array}$$

Suppose it is given an action of $\widehat{\mathcal{W}}$ on $\mathcal{C}$, extending the natural action of $\mathcal{W}$ on $\mathcal{C}$. Then this action induces uniquely an action of $\widehat{N}$ on $\mathcal{A}$, extending the natural action of N on $\mathcal{A}$. There are the following questions:

- Does there exist an action of $\widehat{G}$ on Ω, extending the natural action of G on Ω, such that the diagram commutes?
- If such an action exists, is it uniquely determined?
- Which actions can be obtained in this way?

In Theorem 45 and Corollary 48 of [15] we obtained the following beautiful result: There is a bijective correspondence between:

(i) The actions of $\widehat{\mathcal{W}}$ on $\mathcal{C}$, extending the natural action of $\mathcal{W}$ on $\mathcal{C}$, satisfying certain conditions which we do not state here.
(ii) The actions of $\widehat{G}$ on Ω, extending the natural action of G on Ω.

Furthermore, the action of $\widehat{G}$ on Ω is obtained by an explicit formula from the action of $\widehat{\mathcal{W}}$ on $\mathcal{C}$.

There exist quite general actions of $\widehat{G}$ on Ω, extending the natural action of G on Ω. Compare for example the action of Remark 46 in [15]. There is the following question:

- Is it possible to single out some actions of $\widehat{G}$ on Ω, which extend the natural action of G on Ω and also keep some of its properties?