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A transversality theorem for some classical varieties

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ABSTRACT

In 2009, de Fernex and Hacon [10] proposed a generalization of the notion of the singularities to normal varieties that are not $\mathbb{Q}$-Gorenstein. Based on their work, we generalize Kleiman's transversality theorem to subvarieties with log terminal or log canonical singularities. We also show that some classical varieties, such as generic determinantal varieties, W_d^r for general smooth curves, and certain Schubert varieties in $G(k, n)$ are log terminal in de Fernex and Hacon's notion, and canonical with some suitable boundary in the classical sense.

1. Introduction

In higher dimensional birational geometry, one studies pair (X, Δ) consisting of a variety X and a boundary divisor Δ. The boundary, Δ, plays several useful roles, such as an effective divisor to make the canonical divisor $\mathbb{Q}$-Cartier, or a tool to apply the adjunction formula to do induction. However, in many cases there is not a canonical choice for Δ. In [10], de Fernex and Hacon propose a new approach to birational geometry. Instead of using the boundary divisor, they propose a more direct way to pull back non $\mathbb{Q}$-Cartier divisors. They define a relative canonical divisor which generalizes the classical one, and then extend the singularity theory to non $\mathbb{Q}$-Gorenstein varieties.

They show that if X is log terminal (resp. log canonical) in their sense, then there exists a boundary Δ such that (X, Δ) is Kawamata log terminal (resp. log canonical) in the old sense. On the other hand, Kleiman's theorem plays a central role in the study of intersection theory and enumerative geometry. In this paper, based on de Fernex and Hacon's work we show the following theorem.

Theorem 1.1 (*Kleiman's Transversality Theorem*)**.** *Let X be a homogeneous variety with a group variety G acting on it. Let Y, Z be subvarieties of X such that Z is smooth and Y is log canonical (resp. log terminal). For any $g \in G$, denote Y^g the transition of Y by g. Then there exists a nonempty open $U \subset G$ such that $\forall g \in U$, $Y^g \times_X Z$ is log canonical (resp. log terminal).*

For the original version of Kleiman's Transversality theorem, see [16] or [13] Theorem III.10.8.

There are some classical varieties that are well known to be normal but not $\mathbb{Q}$-Gorenstein. To understand their singularities, one way is to find out a suitable boundary. However, for many of these varieties there is not a natural choice of the boundary divisor . Using the generalized notion of singularity, we can study the singularities in a more direct way. In this paper, we give a criterion for a normal variety to be log terminal.

Theorem 1.2. *Let $f : Y \to X$ be a small resolution of a normal variety X, such that Y is smooth and $-K_Y$ is relatively nef. Then X is log terminal (in the sense of de Fernex and Hacon). Moreover, there is a boundary Δ such that (X, Δ) is canonical (in the old sense).*

Here small means the exceptional locus is of codimension bigger than two. Then we apply this theorem to some classical varieties such as generic determinantal varieties, $W_d^r(C)$ for general smooth curve C, and certain Schubert varieties in $G(k, n)$. For each of them, we construct a small resolution with a relatively nef anti-canonical divisor. Then we conclude that they all

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have log terminal singularities. Besides, there are boundary divisors such that these varieties have canonical singularities as pairs. In particular, they have rational singularities.

The organization of this paper is as follows. In the second section, we recall some notions from [10] and also prove some propositions which will be used in the proofs of the main theorems. The proofs of the main theorems are in Section 3. Examples of log terminal varieties are given in the last three sections.

To simplify the notation, we will denote log canonical (resp. log terminal) by lc (resp. lt). If we want to emphasize that it is the old notion, we will say lc (resp. lt) pair. Throughout the paper, the ground field is the complex numbers.

2. Preliminaries

In this section, we recall some definitions and prove some lemmas that will be used in the proofs of the main theorems. First, we would like to generalize the notion of relative canonical divisor to normal varieties. The idea and notation will be mainly based on the paper [10].

Definition 2.1. Let $f : Y \to X$ be a morphism between two normal varieties. Since Y is normal, for any prime divisor E there is a valuation $V_E(\cdot)$ defined by the local ring of E. For any ideal sheaf $\mathcal{I}$ on Y, the valuation $Val_E(\mathcal{I})$ is defined as

$$Val_E(\mathcal{I}) := min\{V_E(\phi) \mid \phi \in \mathcal{I}(U), U \cap E \neq \emptyset\}.$$

For any Weil divisor D on X, we define the natural pull back as

$$f^{\natural}(D) := div(\mathcal{O}_X(-D) \cdot \mathcal{O}_Y) := \sum_{E \subset Y} Val_E(\mathcal{O}_X(-D) \cdot \mathcal{O}_Y).$$

One of the properties of the natural pull back is that

$$\mathcal{O}_Y(-f^{\natural}D) = (\mathcal{O}_X(-D) \cdot \mathcal{O}_Y)^{\vee\vee}.$$

The natural pull back usually does not have the homogeneity property, i.e, $f^{\natural}(mK_X) \neq mf^{\natural}(K_X)$ (see [10] for example). As a result, instead of defining the relative canonical divisor directly, we have the following:

Definition 2.2. The *m-th limiting relative canonical* $\mathbb{Q}$ *divisor* $K_{m,Y/X}$ is

$$K_{m,Y/X} := K_Y - \frac{1}{m} f^{\natural}(mK_X).$$

Given any prime divisor F on Y, we define the *m-th limiting discrepancy* of X to be

$$a_{m,F}(X) := \mathrm{ord}_F(K_{m,Y/X}).$$

X is said to be lc (resp. lt) if there is some positive integer m_0 such that $a_{m_0,F}(X) \geq -1$ (*resp.* > -1) for every prime divisor F over X.

Let $g : V \to Y$ and $f : Y \to X$ be two birational morphisms, usually we do not expect that $(fg)^{\natural}(mK_X) = g^{\natural}(f^{\natural}mK_X)$. However, the equality holds when $\mathcal{O}_X(-mK_X) \cdot \mathcal{O}_Y$ is an invertible sheaf (Lemma 2.7 in [10]). A consequence of this property is the following lemma:

Lemma 2.3 (*Lemma 3.5 in [10]*)**.** *Let m be a positive integer, and let $f : Y \to X$ be a proper birational morphism from a normal variety Y such that mK_Y is Cartier and $\mathcal{O}_X(-mK_X) \cdot \mathcal{O}_Y$ is invertible. Then for every birational morphism $g : V \to Y$ we have*

$$K_{m,V/X} = K_{m,V/Y} + g^* K_{m,Y/X}.$$

In particular, when we study the singularities of a given variety X, it suffices to find a log resolution of $(X, \mathcal{O}_X(-mK_X))$. More precisely, we just need to find a proper birational morphism $f : Y \to X$ such that Y is smooth and $\mathcal{O}_X(-mK_X) \cdot \mathcal{O}_Y$ is the invertible sheaf of a divisor F, and the exceptional locus is a divisor E such that $F \cup E$ is simple normal crossing. The reason is that for any divisor E over X, by taking a common resolution, we can assume E is on a higher resolution $g : V \to Y$. Then since Y is smooth, the order of $K_{m,V/Y}$ over E must be nonnegative, hence will not affect the type of singularities of X.

Moreover, for any $m \geq 2$, we can find an *m-compatible* boundary Δ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and

$$K^{\Delta}_{Y/X} := K_Y + \Delta_Y - f^*(K_X + \Delta) = K_{m,Y/X}. \tag{1}$$

See [10] for the definition of an m-compatible boundary. We will give a proof of a similar result in Lemma 2.9.

So if X is log terminal (resp, log canonical) in the new notion, there is a boundary Δ (m-compatible for some positive integer m) such that (X, Δ) is Kawamata log terminal (resp, log canonical) in the classical sense. On the other hand, for any boundary Δ such that $m(K_X + \Delta)$ is Cartier, we have $K^{\Delta}_{Y/X} \leq K_{m,Y/X}$ (Remark 3.9 in [10]). As a result, the following concludes.

Proposition 2.4 (*Proposition 7.2 in [10]*)**.** *X is log canonical (resp. log terminal) if and only if there is a boundary Δ such that (X, Δ) is a log canonical pair (resp. log terminal pair).*