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Some good-filtration subgroups of simple algebraic groups

Chuck Hague [a,*], George McNinch [b]

[a] Department of Mathematical Sciences, University of Delaware, 501 Ewing Hall, Newark, DE 19716, United States

[b] Department of Mathematics, Tufts University, 503 Boston Ave, Medford, MA 02155, United States

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ABSTRACT

Let G be a connected and reductive algebraic group over an algebraically closed field of characteristic $p > 0$. An interesting class of representations of G consists of those G-modules having a *good filtration* — i.e. a filtration whose layers are the induced highest weight modules obtained as the space of global sections of G-linearized line bundles on the flag variety of G. Let $H \subset G$ be a connected and reductive subgroup of G. One says that (G, H) is a *Donkin pair*, or that H is a *good filtration subgroup* of G, if whenever the G-module V has a good filtration, the H-module $res^G_H V$ has a good filtration.

In this paper, we show when G is a "classical group" that the *optimal* SL_2-subgroups of G are good filtration subgroups. We also consider the cases of subsystem subgroups in all types and determine some primes for which they are good filtration subgroups.

1. Introduction

Let G be a connected and reductive group over an algebraically closed field of characteristic $p > 0$, and let $H \subset G$ be a closed subgroup which is also connected and reductive. We are concerned here with the linear representations of the algebraic group G – i.e. with G-modules – and with their restriction to H.

Of particular interest are the induced G-modules $\nabla_G(\lambda)$ and the induced H-modules $\nabla_H(\lambda)$ obtained as global sections of equivariant bundles on the associated flag varieties; see Section 2.1. One says that H is a *good-filtration subgroup* of G – or that (G, H) is a *Donkin pair* – provided that for any induced G-module V the H-module $res^G_H V$ obtained from V by restriction to H has an exhaustive filtration whose successive quotients are induced H-modules.

Donkin proved in [7] – under some mild assumptions on the characteristic – that a Levi factor of a parabolic subgroup of G is always a good-filtration subgroup; subsequently, Mathieu gave an unconditional proof [14] of this result using the geometric method of Frobenius splitting (cf. also the accounts in [3] Section 4, [11] Ch. G, and [22]).

In [4], Brundan proved that a large class of reductive spherical subgroups of G are good filtration subgroups, under mild restrictions on p; recall that a subgroup H is said to be *spherical* if there is a dense H-orbit on the flag variety G/B of G. In that paper, Brundan also conjectured that H is a good filtration subgroup if either (i) H is the centralizer of a graph automorphism of G, or (ii) H is the centralizer of an involution of G and $p > 2$. Brundan's conjecture is now a theorem; many cases were covered already in [4] and the remaining cases were handled by van der Kallen in [21].

In this paper we extend the study of Donkin pairs to more reductive subgroups of G. In particular, we consider two classes of reductive subgroups: optimal SL_2-subgroups and the so-called subsystem subgroups. In Section 2 we give preliminaries on algebraic groups, good filtrations, and optimal SL_2 subgroups.

In this paper a group of *classical type*, or just a *classical group*, will be a group isomorphic to $SL(V)$ or the stabilizer of a nondegenerate alternating or bilinear form β when $p > 2$. In Section 3 we give a general criterion for a reductive subgroup

* Corresponding author.
E-mail addresses: hague@math.udel.edu (C. Hague), george.mcninch@tufts.edu (G. McNinch).
URLs: http://sites.google.com/site/chuckhague/ (C. Hague), http://gmcninch.math.tufts.edu/ (G. McNinch).

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of a group of classical type to be a good filtration subgroup (3.2.6). Since a group of classical type is not simply-connected when the form β is symmetric we also consider the simply-connected covers of these groups (3.2.7). We then give a criterion for checking when a reductive subgroup of a group of exceptional type is a good filtration subgroup (Theorem 3.6.3).

In Section 4 we give our main results. In Section 4.1 we consider the case in which G is a classical group and $S \subset G$ is an *optimal* SL_2 *subgroup*, a notion essentially due to Seitz [20]; we follow the characterization of these subgroups given in [15]. The main result of this section is Theorem 4.1.2, which states that optimal SL_2-subgroups of classical groups are good filtration subgroups. Our proof is modeled on arguments of Donkin from [7]. We also consider optimal SL_2 subgroups of the simply-connected covers of classical groups (Theorem 4.1.4). In Section 4.2 we consider optimal SL_2 subgroups of exceptional groups. In these theorems we crucially use induction arguments which reduce to the case of a *distinguished* optimal SL_2 subgroup.

Recall that a *subsystem subgroup* of G is a connected semisimple subgroup which is normalized by a maximal torus. In Section 4.3 we consider arbitrary subsystem subgroups of semisimple groups. Since Brundan's conjecture implies that every subsystem subgroup of a group of type A, B, C, or D is a good filtration subgroup when $p > 2$, we only consider the exceptional case. By the transitivity of the good filtration subgroup property and the fact that Levi factors of parabolic subgroups are good filtration subgroups, it suffices to consider only the case where the subsystem subgroup is of maximal rank (=rank G). The main result in this section is Theorem 4.3.3, which gives primes p for which the maximal rank reductive subgroups not already covered by the Brundan conjecture are good filtration subgroups.

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2. Preliminaries

2.1. Induced modules for reductive groups

Let k be an algebraically closed field of positive characteristic p and let G be a connected and reductive algebraic group over k. Fix a maximal torus $T \subset G$, and choose a Borel subgroup $B \subset G$ containing T. For us, a representation of a linear algebraic group always means a rational representation; namely, a co-module for the coordinate algebra.

We write $X^*(T)$ for the character group and $X_*(T)$ for the co-character group of the torus T. We write $(\lambda, \phi) \mapsto \langle \lambda, \phi \rangle \in \mathbf{Z}$ for the natural pairing $X^*(T) \times X_*(T) \to \mathbf{Z}$. Recall that the choice of the Borel subgroup B determines a system of positive roots R^+ of the set of roots $R \subset X^*(T)$.

Each character $\lambda \in X^*(T)$ determines a G-linearized line bundle $\mathscr{L}(\lambda)$ on the flag variety G/B. The group G acts linearly on the space of global sections

$$H^0(G/B, \mathscr{L}(\lambda));$$

we write $\nabla_G(\lambda)$ for this G-module (which is denoted $H^0_G(\lambda) = H^0(\lambda)$ in [11] Section II). Then $\nabla_G(\lambda)$ is non-0 if and only if λ is *dominant*; i.e. if and only if $\langle \lambda, \alpha^\vee \rangle \geq 0$ for each $\alpha \in R^+$. The representations $\nabla_G(\lambda)$ are known as *induced modules* for G.

Assume that G is quasisimple; in this case, we number the nodes of the Dynkin diagram of G – and hence the simple roots and fundamental dominant weights – as in Bourbaki [2], Plate I–IX. Let $\varpi_i \in X^*(G) \otimes \mathbf{Q}$ denote the *fundamental dominant weights*; if $\alpha_1, \ldots, \alpha_r$ are the simple roots with corresponding co-roots $\alpha_i^\vee \in X_*(T)$, then $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{ij}$ for $1 \leq i, j \leq r$. Of course, G is *simply connected* if and only if $\varpi_i \in X^*(T)$ for $1 \leq i \leq r$.

2.2. Modules with a good filtration

Let V be any G-module. A collection of G-submodules $V_i \subset V$ for $i \in \mathbf{Z}_{\geq 0}$ forms a *filtration* of V provided that $V_i \subset V_{i+1}$ for $i \geq 0$ and that $V = \bigcup_{i \geq 0} V_i$. The *layers* of the filtration are the quotient modules V_i/V_{i-1}.

The filtration of V is said to be a *good filtration* if for each $i \geq 1$, the layer V_i/V_{i-1} is either 0 or is isomorphic to an induced module $\nabla_G(\lambda_i)$ for some dominant weight λ_i.

For a G-module V with a good filtration, the *support* of V (written as *Supp*(V)) is the set of $\lambda \in X_G^+$ for which $\nabla_G(\lambda)$ occurs as a layer in a good filtration of V. It follows from [11], Prop. II.4.16 that the support of V is independent of the choice of good filtration of V.

2.2.1. *Let* $(*) \quad 0 \to V \to E \to W \to 0$ *be a short exact sequence of G-modules.*

(a) *Assume that V has a good filtration. Then E has a good filtration if and only if W has a good filtration.*
(b) *If the sequence* $(*)$ *is* split exact, *and if E has a good filtration, then both V and W have a good filtration.*

Proof. Assertion (a) follows from the "homological" characterization of good filtrations found in [11], Prop. II.4.16, and (b) is an immediate consequence of (a). □

We also observe the following:

2.2.2. *If the G-module V has a filtration for which each quotient* V_i/V_{i-1} *has a good filtration for* $i \geq 1$, *then V has a good filtration.*

Proof. This is straightforward when V is finite dimensional; the general case is obtained in [7], Prop. 3.1.1. □