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Strong nonnegativity and sums of squares on real varieties

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ABSTRACT

Motivated by scheme theory, we introduce strong nonnegativity on real varieties, which has the property that a sum of squares is strongly nonnegative. We show that this algebraic property is equivalent to nonnegativity for nonsingular real varieties. Moreover, for singular varieties, we re-prove and generalize obstructions of Gouveia and Netzer to the convergence of the theta body hierarchy of convex bodies approximating the convex hull of a real variety.

1. Introduction

The relationship between nonnegative polynomials and sums of squares of polynomials on real varieties is a classical subject, dating back to Hilbert. In real algebraic geometry, a large body of research is dedicated to understanding the gap between these families. At the same time, this subject has recently become important in the emerging field of convex algebraic geometry, where it is relevant to the effectiveness of computing convex hulls of algebraic varieties. This in turn has been intimately related to the geometry of feasible regions of semidefinite programs (see [6] and references therein). Motivated by this and inspired by scheme theory, we introduce an intermediate class of polynomials which we call *strongly nonnegative*. This class is particularly useful for understanding the role that singularities on real varieties play in obstructing sums of squares representations.

We begin by exploring the basic properties of strong nonnegativity, showing in particular in Theorem 2.10 that strong nonnegativity at a point implies nonnegativity in a neighborhood of that point, and that the converse holds for nonsingular points. In the singular case, we study obstructions to the theta body hierarchy [5] of convex bodies approximating the convex hull of a real variety. The strength of this approximation is governed by the sums of squares representability of linear functions on a variety. We are able to recover very transparently in Theorem 4.4 the obstructions produced by Gouveia and Netzer in [4] to convergence of this hierarchy. The same argument gives us Corollary 4.3, a generalized version of their obstruction. Finally, Proposition 4.7 shows that our construction behaves well in the context of the foundational constructions of Gouveia et al. in [5].

2. Strong nonnegativity

Our convention throughout, given an ideal $I \subseteq \mathbb{R}[x_1, \ldots, x_n]$, is to use $V_{\mathbb{R}}(I)$ for the real vanishing set of I, and use $V(I)$ in relation to concepts depending on the ring $\mathbb{R}[x_1, \ldots, x_n]/I$, which we will denote by A. Formally, $V(I)$ is the closed subscheme $\mathrm{Spec}(A) \subseteq \mathbb{A}^n_{\mathbb{R}}$, but our definitions will be in terms of A, so no knowledge of schemes is required. All of our ring homomorphisms are assumed to be $\mathbb{R}$-algebra homomorphisms.

We begin by introducing our stricter definition of nonnegativity. Our motivating example is the following (see Fig. 1):

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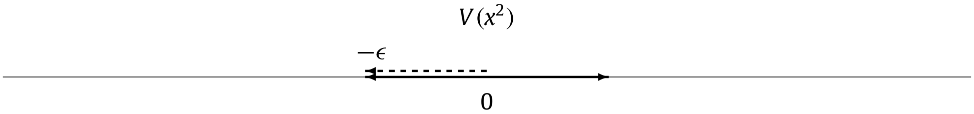

Fig. 1. Motivating example.

Example 2.1. Suppose $I \subseteq \mathbb{R}[x]$ is the ideal generated by x^2. Then set-theoretically, we have $V_{\mathbb{R}}(I)$ equal to the origin. Thus, the function x is nonnegative on $V_{\mathbb{R}}(I)$. However, one easily checks that x is not a sum of squares modulo I.

From a more scheme-theoretic perspective, we should think of $V(I)$ as not consisting only of the origin, but also including an infinitesimal thickening in both directions – in particular, in the negative direction. Thus, we should not think of x as being nonnegative on the *scheme* $V(I)$.

Recall that if $I \subseteq \mathbb{R}[x_1, \dots, x_n]$ is an ideal, then the points of $V_{\mathbb{R}}(I)$ correspond precisely to ($\mathbb{R}$-algebra) homomorphisms $A \to \mathbb{R}$, where $A = \mathbb{R}[x_1, \dots, x_n]/I$. The homomorphism obtained from a given $P \in V_{\mathbb{R}}(I)$ is simply given by evaluating polynomials at P. Thus, one may rephrase nonnegativity as saying that f is nonnegative if its image under any homomorphism $A \to \mathbb{R}$ is nonnegative. Our definition will consider a broader collection of such homomorphisms. In particular, given a point of $V_{\mathbb{R}}(I)$ corresponding to $\varphi : A \to \mathbb{R}$, it is standard that the (scheme-theoretic) tangent space of $V(I)$ at the point is in bijection with homomorphisms $A \to \mathbb{R}[\epsilon]/(\epsilon^2)$ which recover φ after composing with the unique homomorphism $\mathbb{R}[\epsilon]/(\epsilon^2) \to \mathbb{R}$, which necessarily sends ϵ to 0.

In Example 2.1, a tangent vector in the "negative direction" is given by the homomorphism $\mathbb{R}[x]/(x^2) \to \mathbb{R}[\epsilon]/(\epsilon^2)$ sending x to $-\epsilon$. If we consider $-\epsilon$ to be "negative", we may thus consider the function x to take a negative value on this tangent vector of $V(I)$. We formalize and generalize this idea by considering also higher-order infinitesimal arcs, as follows.

Definition 2.2. Given $f \in \mathbb{R}[\epsilon]/(\epsilon^m)$, $f = a_0 + a_1\epsilon + \cdots + a_{m-1}\epsilon^{m-1}$, we say that f is **nonnegative** if $f = 0$, or $a_N > 0$ where $N = \min\{j : a_j \neq 0\}$.

Note that $\mathbb{R}[\epsilon]/(\epsilon^m)$ has a unique homomorphism to $\mathbb{R}$, necessarily sending ϵ to 0. We say that $\varphi : A \to \mathbb{R}[\epsilon]/(\epsilon^m)$ is **at** P for (a necessarily unique) $P \in V_{\mathbb{R}}(I)$ if P is the point corresponding to the composed homomorphism $A \to \mathbb{R}$.

Definition 2.3. Let $I \subseteq \mathbb{R}[x_1, \dots, x_n]$ be an ideal, and $A := \mathbb{R}[x_1, \dots, x_n]/I$. Given $P \in V_{\mathbb{R}}(I)$, we say $f \in A$ is **strongly nonnegative** at P if for every $m \geq 0$ and for every $\mathbb{R}$-algebra homomorphism

$$\varphi : A \to \mathbb{R}[\epsilon]/(\epsilon^m)$$

at P, we have that $\varphi(f)$ is nonnegative. We say that f is **strongly nonnegative** on $V(I)$ if it is strongly nonnegative at P for all $P \in V_{\mathbb{R}}(I)$.

One might naturally wonder whether we can make a stronger version of strong nonnegativity by considering homomorphisms to rings such as $\mathbb{R}[\epsilon_1, \dots, \epsilon_r]/(\epsilon_1, \dots, \epsilon_r)^m$, and defining nonnegativity in terms of a suitable monomial order. This does not behave well for the lexicographic order, since the lowest term in this order may have the highest degree, so (sums of) squares need not be nonnegative. In contrast, for the graded lexicographic order, Proposition 2.13 below shows that we do not get anything new, and a similar construction works also for the (graded) reverse lexicographic order.

We begin with some basic observations on the property of strong nonnegativity.

Proposition 2.4. *Given $f \in A$, we have the following statements.*

(1) *If f is strongly nonnegative at $P \in V_{\mathbb{R}}(I)$, then f is nonnegative at P.*
(2) *If f is strictly positive at $P \in V_{\mathbb{R}}(I)$, then f is strongly nonnegative at P.*
(3) *If f is a sum of squares, then f is strongly nonnegative.*

Proof. We obtain (1) immediately by setting $m = 1$ in the definition, since this yields the evaluation map at P.

For (2), given any homomorphism $\varphi : A \to \mathbb{R}[\epsilon]/(\epsilon^m)$ at P, by definition we have that composing with $\mathbb{R}[\epsilon]/(\epsilon^m) \to \mathbb{R}$ gives the evaluation map at P, under which f is strictly positive by hypothesis. But then if we write $\varphi(f) = a_0 + a_1\epsilon + \cdots + a_{n-1}\epsilon^{n-1}$, we must have $a_0 = f(P) > 0$, and thus $\varphi(f)$ is nonnegative. Since φ was arbitrary at P, we conclude that f is strongly nonnegative at P.

Finally, for (3), if $f = \sum_{i=1}^r h_i^2$, and $\varphi : A \to \mathbb{R}[\epsilon]/(\epsilon^m)$ is an $\mathbb{R}$-algebra homomorphism, then the leading term of each $(\varphi(h_i))^2$ is nonnegative, and hence so is that of $\varphi(f)$. □

We will show in Theorem 2.10 that in fact if f is strongly nonnegative at P, then it is nonnegative on a neighborhood of P, and that the converse holds if P is a nonsingular point of $V(I)$. Of course, the converse does not hold in general.

Example 2.5. Consider $I = (y - x^2, y^2) \subseteq \mathbb{R}[x, y]$, and $P = (0, 0)$ the only point of $V_{\mathbb{R}}(I)$. Then $-y$ is not strongly nonnegative on $V(I)$: under the homomorphism $\varphi : \mathbb{R}[x, y]/I \to \mathbb{R}[\epsilon]/(\epsilon^3)$ at P sending x to ϵ and y to ϵ^2, we have that $\varphi(-y) = -\epsilon^2$ is not nonnegative (see Fig. 2).

On the other hand, y is strongly nonnegative on $V(I)$ by Proposition 2.4 (3), since $y = x^2$ modulo I.

We also give an example where $V(I)$ is reduced (i.e., I is radical) for which strong nonnegativity is strictly stronger than nonnegativity.