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A monotone geometric mean for a class of Toeplitz matrices

Elham Nobari

Department of Mathematics, University of Science and Technology of Mazandaran, Behshahr, Iran

A R T I C L E I N F O

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A B S T R A C T

In this paper, using Laurent operators and Fourier coefficients of their symbol functions, we introduce a geometric mean for a large class of $n \times n$ positive semi-definite Toeplitz matrices which satisfies the monotonicity property. The cost of our approach in term of arithmetic operations for m matrices is of the order $O(mn^2)$. This definition preserves the structure, is simple to calculate, preserves monotonicity and satisfies some other Ando–Li–Mathias properties.

1. Introduction

The notion of geometric mean for positive definite matrices naturally appears in several areas, for instance in radar detection [12,16], image processing [15] and elasticity tensor analysis [14]. A definition of geometric mean of three or more positive definite matrices has been defined by M. Moakher [13] and R. Bhatia, J. Holbrook [4,5]. This is

E-mail address: enobary@gmail.com.

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usually identified with Karcher mean. T. Ando, C. Li and R. Mathias in [1] have introduced a definition of geometric mean and have shown some of its properties, called the Ando–Li–Mathias (ALM) axioms. These properties should be required for any reasonable notion of geometric mean of the matrices. There is a rich literature of the geometric means of matrices and methods for computing them, see [6,7,11] and the references in [7,11]. But there are also many unsolved problems in this field yet. The Karcher mean does not preserve structures, for example the Karcher mean of two Toeplitz matrices is not necessarily a Toeplitz matrix. D.A. Bini et al. in [7] have introduced a definition of geometric mean for structured matrices. This definition satisfies many of the ALM properties except monotonicity. Moreover, this method can not guarantee uniqueness of the structured geometric mean.

In this paper, we consider only the Toeplitz matrices. A Toeplitz matrix is a matrix in which entries along their diagonals are constant. These matrices have many applications in a wide variety of problems in engineering. For positive definite Toeplitz matrices, there is the interesting notion of mean based on Kähler metric [2,3] which is not a geometric mean, but satisfies some desirable properties such as permutation invariance and repetition invariance. This mean, called Kähler metric mean, does not coincide with Karcher mean but with this manner the mean of two positive definite Toeplitz matrices will be a Toeplitz matrix again. Unfortunately, the Kähler metric mean is not monotonic.

In this article we introduce a new definition of geometric mean for a class of Toeplitz matrices. This approach is an operator theoretical approach as follows. For every $n \times n$ Toeplitz matrix

$$A = \begin{bmatrix} a_0 & a_1 & \cdots & & a_{n-1} \\ a_{-1} & a_0 & a_1 & \cdots & \\ \vdots & & \ddots & & \vdots \\ & & \ddots & & a_1 \\ a_{-n+1} & & \cdots & a_{-1} & a_0 \end{bmatrix},$$

we consider the function $a : \mathbb{T} \longrightarrow \mathbb{C}$, where $\mathbb{T} = \{t \in \mathbb{C} : |t| = 1\}$ is the unit circle in the plane by definition $a(t) = \sum_{k=-n+1}^{n-1} a_k t^k$ for all $t \in \mathbb{T}$. Now, let $M(a) \in \mathcal{L}\big(L^\infty(\mathbb{T})\big)$ be the multiplication operator associated to the function a, i.e., $M(a)f = af$ for all $f \in L^\infty(\mathbb{T})$. In fact $M(a)$ is a Laurent operator with the so-called ‘symbol’ function a, see [10] and (3.1) below. We denote the cone of all positive semi-definite $n \times n$ Toeplitz matrices with non-negative symbols by $\mathcal{T}_n^{++}$, see (3.4) and the Lemma 3.5. In this paper we introduce a new definition of geometric mean on $\mathcal{T}_n^{++}$ which satisfies among other properties, the monotonicity property in the ordering induced by the cone $\mathcal{T}_n^{++}$, see (3.5) and Theorem 3.7. Comparing to the other approaches, our proposed definition admits some important advantages: low cost in terms of arithmetic operations, simple calculations, structure preserving and monotonicity. Moreover, we do not use the non-singularity of matrices, e.g., the zero matrix belongs to $\mathcal{T}_n^{++}$.