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The Hua matrix and inequalities related to contractive matrices

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A R T I C L E I N F O

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We first deny a conjecture raised in Xu et al. (2011) [14] and then we present some eigenvalue or singular value inequalities related to contractive matrices.

1. Introduction

The Hua matrix is

$$\mathbf{H} = \begin{bmatrix} (I - A^*A)^{-1} & (I - B^*A)^{-1} \\ (I - A^*B)^{-1} & (I - B^*B)^{-1} \end{bmatrix},$$

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where A, B are $n \times n$ strictly contractive matrices (i.e., matrices whose spectral norm is less than one). This block matrix first appears in Hua's study of the theory of functions of several complex variables; see [9]. The Hua matrix is a source for matrix inequalities. For example, the positivity of the Hua matrix immediately leads to

$$|\det(I - A^*B)|^2 \geq \det(I - A^*A)\det(I - B^*B), \tag{1.1}$$

which is known as Hua's determinantal inequality in the literature (e.g., [16, p. 231]). More examples can be found in [3,4,12]. There is a renewed interest in the Hua matrix and its analogues in recent years; see [2,10,11,13,14]. A remarkable property about the Hua matrix is the positive partial transpose property. That is, the partial transpose of $\mathbf{H}$, viz.,

$$\mathbf{H}^\tau = \begin{bmatrix} (I - A^*A)^{-1} & (I - A^*B)^{-1} \\ (I - B^*A)^{-1} & (I - B^*B)^{-1} \end{bmatrix}$$

is again positive semidefinite.

In this article, we first address a conjecture raised in [14], then we present some eigenvalue or singular value inequalities involving contractive matrices. The paper is concluded with some comments and a new conjecture along this line of study.

The remaining of this section is devoted to some notation used in this article. Let $\mathbb{M}_n$ be the set of all $n \times n$ complex matrices; the identity matrix of $\mathbb{M}_n$ is denoted by I. For any $X \in \mathbb{M}_n$, X^* stands for the conjugate transpose of X. For two Hermitian matrices X, Y of the same size, we write $X \geq Y$ to mean $X - Y$ is positive semidefinite. Saying that $X \in \mathbb{M}_n$ is contractive is the same as saying $I \geq X^*X$. If the eigenvalues of a square matrix X are all real, then we denote $\lambda_j(X)$ the jth largest eigenvalue of X. The singular values of a complex matrix X are the eigenvalues of $|X| := (X^*X)^{1/2}$, and we denote $\sigma_j(X) := \lambda_j(|X|)$. The geometric mean of two positive definite matrices $X, Y \in \mathbb{M}_n$ is defined as $X \sharp Y := X^{1/2}(X^{-1/2}YX^{-1/2})^{1/2}X^{1/2}$. Its weighted version is defined as $X \sharp_t Y := X^{1/2}(X^{-1/2}YX^{-1/2})^t X^{1/2}$, $0 \leq t \leq 1$. It is known that the notion of (weighted) geometric mean could be extended to cover all positive semidefinite matrices; see [5, p. 107].

2. A conjecture in [14]

When considering possible extensions of the Hua matrix to higher number of blocks, it is known (see [2,13]) that in general the following block matrices are no longer positive semidefinite for $m \geq 3$:

$$\mathbf{H}_{(m)} = \begin{bmatrix} (I - A_1^*A_1)^{-1} & (I - A_2^*A_1)^{-1} & \cdots & (I - A_m^*A_1)^{-1} \\ (I - A_1^*A_2)^{-1} & (I - A_2^*A_2)^{-1} & \cdots & (I - A_m^*A_2)^{-1} \\ \vdots & \vdots & & \vdots \\ (I - A_1^*A_m)^{-1} & (I - A_2^*A_m)^{-1} & \cdots & (I - A_m^*A_m)^{-1} \end{bmatrix},$$