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On Hille-type approximation of degenerate semigroups of operators

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ABSTRACT

The result that goes essentially back to Euler [15] says that for any element a of a unital Banach algebra $\mathbb{A}$ with unit u, the limit $\lim_{\varepsilon\to 0+}(u+\varepsilon a)^{[\varepsilon^{-1}t]}$ (where $[\cdot]$ denotes the integral part) exists for all $t\in\mathbb{R}$ and equals e^{ta}. As developed by E. Hille [22, Thm. 12.2.1], in the case where a is replaced by the generator A of a strongly continuous semigroup $\{\mathrm{e}^{tA}, t\ge 0\}$ in a Banach space $\mathbb{X}$, a proper counterpart of this formula is $\mathrm{e}^{tA}=\lim_{\varepsilon\to 0+}(I_{\mathbb{X}}-\varepsilon A)^{-[\varepsilon^{-1}t]}$ strongly in $\mathbb{X}$. Motivated by an example from mathematical biology (related to Rotenberg's model of cell growth [40]) we study convergence of a similar approximation in which u (resp. $I_{\mathbb{X}}$) is replaced by $j\in\mathbb{A}$ (resp. $J\in\mathcal{L}(\mathbb{X})$) such that for some $\ell\ge 2$, $j^\ell=u$ (resp. $J^\ell=I_{\mathbb{X}}$).

1. Introduction

While, with few exceptions, impact of mathematics on biology is still rather disputable (see, however, the recent paper [39] and its predecessor [38]), mathematical biology con-

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tinues to surprise mathematicians by the constant flow of interesting objects to study. For example, the Wright–Fisher model of population genetics (see e.g. [16,17]) has provided proper intuitions for the discovery of the general form of boundary conditions for linear parabolic partial differential equations, known today as Feller–Wentzell boundary conditions [18–20,44]. The same model had a considerable impact on the theory of exchangeability of random processes [27], hiding in particular a mathematical diamond, i.e., the Kingman–Tajima coalescence process [16,26,28,42] (see also the survey article [31]). In fact, any list of mathematical inspirations coming from biology seems to be doomed to be incomplete (see e.g. [1,5,9,16,24,25,34,43]).

The story this paper tells is also of biological origin, though perhaps not as remarkable as the stories just touched upon. To begin, for real numbers a and b, let $L^1(a,b)$ be the space of integrable functions on (a,b). In the Rotenberg's model of cell division [40], popularized by Baulanouar's papers (see [10,11] and other articles cited therein), cells in a population are characterized by maturity parameter $\mu \in [0,1]$ and the speed of maturation v. Thus, in a version of the model in which the set V of possible speeds is finite i.e., $V = \{v_i, i \in \mathcal{N}\}$, where $\mathcal{N} = \{1,\dots,N\}$ for some $N \in \mathbb{N}$, and $\phi_i \in L^1(0,1)$ is the density of cells with speed v_i at time 0, then for $\mu > 0$ and sufficiently small t, this density at time $t > 0$ is

$$\phi_i(t,\mu) = \phi_i(\mu - v_i t).$$

(In the original model [40], V is equal to $[0,\infty)$, but the variant with V finite also leads to interesting theory, linking the model in particular with flows on networks, see [2,4,3,33].) Upon reaching maturity $\mu = 1$, cells divide and each of daughter cells' maturation speeds may differ from their mother's. Moreover, some cells degenerate, and come back to the state $\mu = 0$ while retaining their maturation speed. A balance condition 'flux in equals flux out' says therefore that

$$v_i\phi_i(t,0) = p\sum_{j\neq i} \pi_{ji} v_j \phi_j(t,1) + q v_i \phi_i(t,1),$$

where $p > 0$ is the average number of viable cells after division, $q \in [0,1)$ is the number of degenerating cells, and π_{ji} is the probability that a cell of maturation speed v_j will have a daughter of maturation speed v_i. This equation, sometimes termed the Lebowitz–Rubinow *boundary* condition [12] (although the adjective *transmission* would be more appropriate here), thus connects the value of the vector $\phi(t) = (\phi_i(t))_{i\in\mathcal{N}} \in [L^1(0,1)]^N$ of densities $\phi_i(t)$ at $\mu = 0$, with this at $\mu = 1$:

$$\phi(t,0) = q\phi(t,1) + pK\phi(t,1), \tag{1}$$

where K is an appropriate $N \times N$ matrix.

In [2,4], Banasiak and Falkiewicz study a singular perturbation of the Rotenberg model in which the velocities become simultaneously infinite, while the cells have an increasing tendency to degenerate. In particular, one may think of $q = 1 - \varepsilon$ and $p = \varepsilon$