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Questions, conjectures, and data about multiplicity lists for trees ☆

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ABSTRACT

We review and discuss a number of questions and conjectures about multiplicity lists occurring among real symmetric matrices whose graph is a tree. Our investigation is aided by a new electronic database containing all multiplicity lists for trees on fewer than 12 vertices. Some questions and conjectures are familiar and some are new, and new information is given about several.

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Eigenvalue
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Tree

1. Introduction

Let G be an undirected graph without loops, and denote by $\mathcal{H}(G)$ the set of all Hermitian matrices whose graph is G. No restriction other than reality is placed upon the diagonal entries of $A \in \mathcal{H}(G)$. Among the matrices in $\mathcal{H}(G)$ are various spectra, and each of these corresponds naturally to a multiplicity list, which we usually consider to be an unordered collection. For example, a classical result is that when G is a path, only the list of all 1's occurs.

Let $\mathcal{L}(G)$ denote the collection of all multiplicity lists among matrices in $\mathcal{H}(G)$. It is a large and combinatorially intriguing problem to predict $\mathcal{L}(G)$ from the structure of G. When G is a tree T, there are several striking relationships between the characteristics of T and the attainable lists in $\mathcal{L}(T)$, that convey more structure than for general graphs (see references), though there is, by no means, a complete answer. In the study of this problem, a number of conjectures and questions have emerged.

Our purpose here is two-fold: (1) to popularize these questions and conjectures, many of which have not previously appeared in print, and (2) to announce the existence of an electronic database currently containing all multiplicity lists for all trees on fewer than 12 vertices. Each question/conjecture is stated and discussed in the following sections, along with prior relevant results and new results, either theoretical or gleaned from the database. The questions/conjectures are organized loosely into 3 sections by whether they give necessary, sufficient or other conditions on lists. The database is searchable, and the lists represent a combination of prior published results along with the 11-vertex trees, which were determined based on some recent results about linear trees [10] as well as some calculations and proofs that we carried out. First, in Section 2, we mention important background. Then we introduce and discuss the database in Section 3, followed by several questions/conjectures grouped into three sections. Appendix A includes a complete inventory of the multiplicity lists of 11-vertex trees.

2. Background and notation

We will use the standard submatrix notation. For an index set $\alpha \subseteq \{1, \dots, n\}$, we denote the principal submatrix of A lying in rows and columns $\{1, \dots, n\} \backslash \alpha$ by $A(\alpha)$, or rows and columns α by $A[\alpha]$. Additionally, we abbreviate $A(\{i\})$ by $A(i)$. If A is a matrix with graph G, we may use a subgraph of G to specify an index set. For example, $A[G]$ is simply the matrix A. For any real number λ, we use $m_A(\lambda)$ to denote the multiplicity of λ as an eigenvalue of the matrix A.

A fundamental fact for our work is the interlacing theorem for Hermitian eigenvalues [2]. An immediate consequence is that for any n-by-n Hermitian matrix A, any real λ, and any $i \in \{1, \dots, n\}$,