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On some properties of three different types of triangular blocked tensors ☆

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ABSTRACT

We define and study three different types of upper (and lower) triangular blocked tensors, which are all generalizations of the triangular blocked matrices, and are also generalizations of reducible or weakly reducible tensors. We study some common properties, as well as different properties of these three types of triangular blocked tensors. We obtain the formulas for the determinants, characteristic polynomials and spectra of the first and second type triangular blocked tensors, and give an example to show that these formulas no longer hold for the third type triangular blocked tensors. We prove that the product of any two $(n_1, \cdots, n_r)$-upper (or lower) triangular blocked tensors of the first or second or third type is still an $(n_1, \cdots, n_r)$-upper (or lower) triangular blocked tensor of the same type. We also prove that, if an $(n_1, \cdots, n_r)$-upper triangular blocked tensor of the first or second or third type has a left k-inverse, then its unique left k-inverse is still an $(n_1, \cdots, n_r)$-upper triangular blocked tensor of all the three types. Also if it has a right k-inverse, then all of its right k-inverses are still $(n_1, \cdots, n_r)$-upper triangular blocked tensors of all the three types. By showing that the left k-inverse (if any) of a weakly irreducible nonsingular M-tensor is a positive tensor, we show that the left k-inverse (if any) of a first or second or third type normal $(n_1, \cdots, n_r)$-upper triangular blocked nonsingular

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M-tensor is an $(n_1, \cdots, n_r)$-upper triangular blocked tensor of all the three types all of whose diagonal blocks are positive tensors. We also show that every order m dimension n tensor is permutation similar to some third type normal upper triangular blocked tensor (all of whose diagonal blocks are irreducible). We give an example to show that this is not true for the first type normal upper triangular blocked tensor.

1. Introduction

In recent years, the study of tensors and the spectra of tensors (and hypergraphs) with their various applications has attracted extensive attention and interest, since the work of L. Qi [14] and L.H. Lim [11] in 2005.

As was in [14], an order m dimension n tensor $\mathbb{A} = (a_{i_1 i_2 \cdots i_m})_{1 \le i_j \le n \ (j=1,\cdots,m)}$ over the complex field $\mathbb{C}$ is a multidimensional array with all entries $a_{i_1 i_2 \cdots i_m} \in \mathbb{C}$ $(i_1, \cdots, i_m \in [n] = \{1, \cdots, n\})$.

In this paper, we define and study three different types of the general $(n_1, \cdots, n_r)$-upper (and lower) triangular blocked tensors.

It is well known that the triangular blocked matrices are very important and useful in the study and applications of matrices. For tensors, Hu et al. [8] gave a determinant formula for the special case $r = 2$ of some type of the $(n_1, \cdots, n_r)$-triangular blocked tensors. Also, Shao et al. [16] defined a type of lower triangular blocked tensors which is essentially equivalent to the second type upper triangular blocked tensors defined in this paper (see Theorem 2.5 of this paper for the proof). Shao et al. [16] also studied some other basic properties of such type triangular blocked tensors.

Recently, Hu, Huang and Qi in [9] introduced and studied the "nonnegative tensor partition" which is also essentially equivalent to the second type triangular blocked tensors defined in this paper (up to a permutation similarity). They obtained that every order m dimension n tensor is permutation similar to such a type of upper triangular blocked tensor each of whose diagonal blocks are weakly irreducible (also see Proposition 1 of [10]). Hu and Qi [10] also used this "nonnegative tensor partition" to study some spectral properties of nonnegative tensors, and obtain a necessary and sufficient condition for a nonnegative tensor to have a positive eigenvector.

In this paper, we first give the definitions of three different types of the general $(n_1, \cdots, n_r)$-upper (and lower) triangular blocked tensors, which are all the natural generalizations of the $(n_1, \cdots, n_r)$-upper (and lower) triangular blocked matrices. Then we study some properties of these three types of triangular blocked tensors.

In some sense, the first and second type upper triangular blocked tensors defined in this paper are generalizations of the weakly reducible tensors (see Definition 1.1 below) which corresponds to the case $r = 2$ (with two diagonal blocks) in Definitions 2.1 and 2.2 up to a permutation similarity (also see Remarks 2.1 and 2.2 in §2), while the third type