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The computation of key properties of Markov chains via perturbations

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Computational procedures for the stationary probability distribution, the group inverse of the Markovian kernel and the mean first passage times of a finite irreducible Markov chain, are developed using perturbations. The derivation of these expressions involves the solution of systems of linear equations and, structurally, inevitably the inverses of matrices. By using a perturbation technique, starting from a simple base where no such derivations are formally required, we update a sequence of matrices, formed by linking the solution procedures via generalised matrix inverses and utilising matrix and vector multiplications. Four different algorithms are given, some modifications are discussed, and numerical comparisons are made using a test example. The derivations are based upon the ideas outlined by Hunter [14].

1. Introduction

In Markov chain theory stationary distributions, mean first passage times and the group inverse provide significant information regarding the behaviour of the chain.

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Let $\{X_n, n \geq 0\}$ be a finite Markov chain (M. C.) with state space $S = \{1, 2, \ldots, m\}$ and transition matrix $P = [p_{ij}]$, where $p_{ij} = P\{X_n = j|X_{n-1} = i\}$ for all $i, j \in S$.

It is well known [3,20], that if the M. C. is *regular* (irreducible and aperiodic) then for all i, j, $\lim_{n\to\infty} p_{ij}^{(n)} = \lim_{n\to\infty} p_j^{(n)} = \pi_j$ where $p_{ij}^{(n)} = P\{X_n = j|X_0 = i\}$, $p_j^{(n)} = P\{X_n = j\}$. The limiting probability of being in state j, π_j, is in fact the "*stationary probability*" of being in state j, in that if $P\{X_0 = j\} = \pi_j$ for all j, then $P\{X_n = j\} = \pi_j$, for all j and $n \geq 0$. An important result is that the stationary distribution $\{\pi_j\}$, $(1 \leq j \leq m)$, exists and is unique for all irreducible M. C.'s, that $\pi_j > 0$ for all j, and satisfies the equations (the *stationary equations*)

$$\pi_j = \sum_{i=1}^{m} \pi_i p_{ij} \quad \text{with } \sum_{i=1}^{m} \pi_j = 1. \tag{1.1}$$

If $\boldsymbol{\pi}^T = (\pi_1, \pi_2, \ldots, \pi_m)$, the stationary probability vector, and $\boldsymbol{e}$ is a column vector of 1's, the stationary equations (1.1) can be expressed as

$$\boldsymbol{\pi}^T(I - P) = \mathbf{0}^T, \quad \text{with } \boldsymbol{\pi}^T\boldsymbol{e} = 1. \tag{1.2}$$

Thus $\boldsymbol{\pi}^T$ can be determined by solving a constrained system of linear equations involving the singular matrix $I - P$ (since each row of P is a discrete distribution, and P is a stochastic matrix with each row sum 1, i.e. $P\boldsymbol{e} = \boldsymbol{e}$).

Let $\Pi = \boldsymbol{e}\boldsymbol{\pi}^T$. In the case of a regular M. C. (finite, irreducible and aperiodic), $\lim_{n\to\infty} P^n = \Pi$, and, in the case of a finite irreducible M. C., $\lim_{n\to\infty} \frac{I+P+P^2+\cdots+P^n}{n} = \Pi$.

Let $T_{ij} = \min[n \geq 1, X_n = j|X_0 = i]$ be the first passage time from state i to state j (first return when $i = j$) and define $m_{ij} = E[T_{ij}|X_0 = i]$ as the mean first passage time from state i to state j (or mean recurrence time of state i when $i = j$). It is well known that for finite irreducible M. C.'s all the m_{ij} are well defined and finite. Let $M = [m_{ij}]$ be the mean first passage time matrix. Let $\delta_{ij} = 1$, when $i = j$ and 0, when $i \neq j$. Let $M_d = [\delta_{ij} m_{ij}]$ be the diagonal matrix formed from the diagonal elements of M, and $E = [1]$ (i.e. all the elements are unity).

It is well known [20] that, for $1 \leq i, j \leq m$,

$$m_{ij} = 1 + \sum_{k\neq j} p_{ik} m_{kj}. \tag{1.3}$$

In particular, the mean recurrence time of state j is given by

$$m_{jj} = 1/\pi_j. \tag{1.4}$$

From (1.3) and (1.4) it follows that M satisfies the matrix equation

$$(I - P)M = E - PM_d, \quad \text{with } M_d = (\Pi_d)^{-1}. \tag{1.5}$$