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Characterization of graphs whose signature equals the number of odd cycles

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ABSTRACT

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The signature $s(G)$ of G is the difference between the number of positive eigenvalues and the number of negative eigenvalues of the adjacency matrix $A(G)$. In [20], it was proved that $-c_1(G) \le s(G) \le c_1(G)$, where $c_1(G)$ denotes the number of odd cycles in G. A problem arises naturally: What graphs have signature attaining the upper bound $c_1(G)$ (resp., the lower bound $-c_1(G)$)? In this paper, we focus our attention on this problem, characterizing graphs G whose signature equals $c_1(G)$ (resp., $-c_1(G)$).

1. Introduction

Throughout this paper we consider simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The adjacency matrix $A(G)$ of G is defined to be a symmetric matrix $[a_{xy}]$ with $a_{xy} = 1$ if $x \in V(G)$ is adjacent to $y \in V(G)$, and $a_{xy} = 0$ otherwise. The eigenvalues of $A(G)$ are said to be the eigenvalues of G, and they form the spec-

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trum of G. The number of positive eigenvalues (resp., negative eigenvalues; resp., zero eigenvalues) of G is called the positive inertia index (resp., the negative inertia index; resp., the nullity) of G, and is denoted by $p(G)$ (resp., $n(G)$; resp., $\eta(G)$). The rank of G, denoted by $r(G)$, refers to the rank of $A(G)$. Obviously, $p(G) + n(G) + \eta(G) = |V(G)|$ and $p(G) + n(G) = r(G)$. We call $p(G) - n(G)$ the signature of G and denote it by $s(G)$.

Motivated by the discovery that the nullity of a graph is related to the stability of the molecule represented by the graph (see [1] or [4] for details), some authors pay a lot of attention to the nullity of graphs and obtain abundant results (see [3–6,9–14,16–19, 21–24]). Recently, partly motivated by these results on nullity of graphs, some authors set foot on describing the positive and the negative inertia index of a graph, and obtained some elementary results for a connected graph with at most three cycles (see [8,20,27]). Here, we particularly mention the paper [20], where the authors posed the following conjecture about the signature $s(G)$ of G.

Conjecture 1.1. (Conjecture, [20]) *Let G be a simple graph. Then $-c_3(G) \leq s(G) \leq c_5(G)$, where $c_3(G)$ denotes the number of cycles in G of length $4k + 3$, and $c_5(G)$ denotes the number of cycles in G of length $4l + 5$ with k, l non-negative integers.*

Till now, the conjecture is confirmed only for some special graphs. In [26], the conjecture was proved for line graphs of simple graphs and power trees. In [25] and [15], the conjecture was proved to be true for graphs with vertex-disjoint cycles and for graphs with edge-disjoint cycles, respectively. Ma et al. [20] proved the conjecture for trees, unicyclic and bicyclic graphs, and they obtained a weaker result as follows.

Proposition 1.2. (Theorem 5.2, [20]) *Let G be a simple graph. Then $-c_1(G) \leq s(G) \leq c_1(G)$, where $c_1(G)$ denotes the number of odd cycles in G.*

Now, a problem arises naturally: What graphs have signature attaining the upper bound $c_1(G)$ (resp., the lower bound $-c_1(G)$)?

In this paper, we are devoted to solving the above problem, characterizing graphs G whose signature equals $c_1(G)$ (resp., $-c_1(G)$).

In Section 2, we introduce some known results, in Section 3 we characterize graphs G whose signature equals $c_1(G)$, and in the last section we characterize graphs G whose signature equals $-c_1(G)$.

2. Notation and some known results

A graph H is called an induced subgraph of G if two vertices of $V(H)$ are adjacent in H if and only if they are adjacent in G. Denote by $G - S$, for $S \subseteq V(G)$, the induced subgraph obtained from G by deleting the vertices in S together with all edges incident to them. Sometimes we use the notation $G - H$ instead of $G - V(H)$ if H is an induced subgraph of G. For an induced subgraph H and a vertex x outside H, the induced