Linear Algebra and its Applications 511 (2016) 274–295

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Preconditioned steepest descent-like methods for symmetric indefinite systems ☆

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ARTICLE INFO

Article history:
Received 10 August 2015
Accepted 9 September 2016
Available online 15 September 2016
Submitted by M. Benzi

MSC:
65F10
65N22
65Y20

Keywords:
Linear system
MINRES
Steepest descent
Convergence
Symmetric indefinite
Preconditioning

ABSTRACT

This paper addresses the question of what exactly is an analogue of the preconditioned steepest descent (PSD) algorithm in the case of a symmetric indefinite system with an SPD preconditioner. We show that a basic PSD-like scheme for an SPD-preconditioned symmetric indefinite system is mathematically equivalent to the restarted PMINRES, where restarts occur after every two steps. A convergence bound is derived. If certain information on the spectrum of the preconditioned system is available, we present a simpler PSD-like algorithm that performs only one-dimensional residual minimization. Our primary goal is to bridge the theoretical gap between optimal (PMINRES) and PSD-like methods for solving symmetric indefinite systems, as well as point out situations where the PSD-like schemes can be used in practice.

☆ Results are partially based on PhD thesis [23] of the first coauthor.

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http://dx.doi.org/10.1016/j.laa.2016.09.011

1. Introduction

The Preconditioned Steepest Descent (PSD) iteration is a well known precursor of the optimal Preconditioned Conjugate Gradient (PCG) algorithm for solving Symmetric Positive Definite (SPD) linear systems. Given a system $Ax = f$ with an SPD matrix A and an SPD preconditioner T the method at each iteration i updates the current approximate solution $x^{(i)}$ as

$$x^{(i+1)} = x^{(i)} + \alpha^{(i)} T(f - Ax^{(i)}), \quad i = 0, 1, \ldots; \tag{1}$$

where the iterative parameter $\alpha^{(i)}$ is chosen to ensure that the new approximation $x^{(i+1)}$ has the smallest, among all vectors of the form $x + \alpha T(f - Ax)$, A-norm of the error $x^{(i+1)} - x$.

The optimality of PCG stems from its ability to construct approximations $x^{(i)}$ that globally minimize the A-norm of the error over an expanding sequence of Krylov subspaces while relying on a short-term recurrence [4,11]. In contrast, the PSD iteration (1) is *locally* optimal, searching for a best approximation $x^{(i+1)}$ only in a single direction, given by the preconditioned residual $T(f - Ax^{(i)})$.

The lack of global optimality in PSD leads to a lower convergence rate. In particular, instead of the asymptotic convergence factor $(\sqrt{\kappa}-1)/(\sqrt{\kappa}+1)$, guaranteed by the optimal PCG, each PSD step is guaranteed to reduce the error A-norm by the factor $(\kappa-1)/(\kappa+1)$, e.g., [4,11], and the error Euclidean norm by the factor $1 - 1/\kappa$, see [12], where κ denotes a spectral condition number of the preconditioned matrix TA. Nevertheless, despite its generally slower convergence, PSD (and even simpler iterations, such as Jacobi or Gauss–Seidel) finds its way to practical applications, due to a reduced amount of memory and computations per iteration [15,13,22].

If the matrix A is symmetric *indefinite*, then an optimal analogue of PCG is given by the preconditioned MINRES (PMINRES) algorithm [16,8].[1] Similar to PCG, PMINRES utilizes a short-term recurrence to achieve optimality with respect to the expanding sequence of the Krylov subspaces [11,9]. However, since A is indefinite, minimization of the error A-norm is no longer feasible. Instead, PMINRES minimizes the T-norm of the residual $f - Ax^{(i+1)}$, where T is a given SPD preconditioner.

The symmetry and positive definiteness of the preconditioner is generally critical for PMINRES. Under this assumption the method is guaranteed to converge, with the convergence bound described in terms of the spectrum

$$\Lambda(TA) = \{\lambda_1 \leq \ldots \leq \lambda_p \leq \lambda_{p+1} \leq \ldots \leq \lambda_n\}$$

[1] PMINRES is mathematically equivalent to preconditioned Orthomin(2) and Orthodir(3) algorithms (e.g., [11]) that can as well be viewed as optimal analogues of PCG for symmetric indefinite systems. However, Orthomin(2) can break down, whereas Orthodir(3) has a higher computational cost compared to PMINRES. Therefore, throughout, we do not discuss these two alternative schemes, and consider only the PMINRES algorithm.