Linear Algebra and its Applications 511 (2016) 460–475

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Strong reciprocal eigenvalue property of a class of weighted graphs

R.B. Bapat [a,1], S.K. Panda [b,*,2], S. Pati [b]

[a] Department of Mathematics and Statistics, ISI Delhi, Delhi 110016, India
[b] Department of Mathematics, IIT Guwahati, Guwahati 781039, India

A R T I C L E I N F O

Article history:
Received 26 August 2015
Accepted 27 September 2016
Available online 29 September 2016
Submitted by R. Brualdi

MSC:
05C50
15A18

Keywords:
Adjacency matrix
Perfect matching
Weighted graph
Property (SR)
Corona

A B S T R A C T

Let $\mathcal{H}$ be the class of connected bipartite graphs G with a unique perfect matching $\mathcal{M}$. For $G \in \mathcal{H}$, let $\mathcal{W}_G$ be the set of weight functions w on the edge set $E(G)$ such that $\mathrm{w}(e) = 1$ for each matching edge and $\mathrm{w}(e) > 0$ for each nonmatching edge. Let G_w denote the weighted graph with $G \in \mathcal{H}$ and $\mathrm{w} \in \mathcal{W}_G$. The graph G_w is said to satisfy the reciprocal eigenvalue property, *property (R)*, if $1/\lambda$ is an eigenvalue of the adjacency matrix $A(G_\mathrm{w})$ whenever λ is an eigenvalue of $A(G_\mathrm{w})$. Moreover, if the multiplicities of the reciprocal eigenvalues are the same, we say G_w has the strong reciprocal eigenvalue property, *property (SR)*. Let $\mathcal{H}_g = \{G \in \mathcal{H} \mid G/\mathcal{M} \text{ is bipartite}\}$, where $G/\mathcal{M}$ is the graph obtained from G by contracting each edge in $\mathcal{M}$ to a vertex. Recently in [12], it was shown that if $G \in \mathcal{H}_g$, then G_w has property (SR) for some $\mathrm{w} \in \mathcal{W}_G$ if and only if G_w has property (SR) for each $\mathrm{w} \in \mathcal{W}_G$ if and only if G is a corona graph (obtained from another graph H by adding a new pendant vertex to each vertex of H).

Now we have the following questions. Is there a graph $G \in \mathcal{H} \setminus \mathcal{H}_g$ such that G_w has property (SR) for each $\mathrm{w} \in \mathcal{W}_G$? Are there graphs $G \in \mathcal{H} \setminus \mathcal{H}_g$ such that G_w never has property (SR), not even for one $\mathrm{w} \in \mathcal{W}_G$? Are there graphs $G \in \mathcal{H}$ such that G_w has property (SR) for some $\mathrm{w} \in \mathcal{W}_G$ but not for all

* Corresponding author.
E-mail addresses: rbb@isid.ac.in (R.B. Bapat), p.swarup@iitg.ernet.in (S.K. Panda), pati@iitg.ernet.in (S. Pati).

[1] The first author acknowledges support from the J. C. Bose Fellowship, Department of Science and Technology, Government of India.
[2] This author's research was supported by the fellowship of IIT Guwahati.

http://dx.doi.org/10.1016/j.laa.2016.09.040

w $\in \mathcal{W}_G$? In this article, we supply answers to these three questions. We also supply a graph class larger than $\mathcal{H}_g$ where for any graph G, if G_{w} has property (SR) for one w $\in \mathcal{W}_G$, then G is a corona graph.

1. Introduction

Let G be a simple undirected finite graph with adjacency matrix $A(G)$. We use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of G, respectively. We use $[i, j]$ to denote an edge between the vertices i and j. A perfect matching in a graph G is a spanning forest whose components are paths on two vertices. A graph G is called *singular* (resp. *nonsingular*) if $A(G)$ is singular (resp. nonsingular).

Throughout this article we consider connected bipartite graphs with unique perfect matchings. Given a molecule, consider its skeletal which can be regarded as a graph (with multiple edges); neglect all the hydrogen atoms and their bonds to the carbon; and finally discard all the double bonds. Thus one finds the simple subgraph induced by only those nodes corresponding to the carbon atoms and it is called a Hückel graph (see Fig. 1). The Hückel graph is used to model the molecular orbital energies of hydrocarbon. It has been shown that many families of Hückel graphs are bipartite graphs with unique perfect matchings (see Yates [13]). This is one of the motivations for considering connected bipartite graphs with unique perfect matchings. The following motivation can be found in Godsil [8]. Suppose G is a bipartite graph with adjacency matrix $A(G)$. In a simple model in Quantum Chemistry, the eigenvalues of $A(G)$ have a physical meaning and so the relation between the graph theoretic properties of G and the eigenvalues of $A(G)$ are of some interest. If G is bipartite then the eigenvalues of $A(G)$ are symmetrically placed about the origin. In the cases of most chemical importance $A(G)$ must be nonsingular.

Important notations. By $\mathcal{H}$ let us denote the class of all connected graphs that are bipartite and which have unique perfect matchings. We shall always use $\mathcal{M}$ to denote the unique perfect matching of a graph under discussion, unless otherwise stated. Furthermore, if $v \in V(G)$, then we use v' to denote the vertex such that $[v, v'] \in \mathcal{M}$. An edge in $\mathcal{M}$ is called *matching* edge and an edge in $\mathcal{M}^c$ is called *nonmatching* edge.

Let G be a simple undirected unweighted graph and G_{w} be the positive weighted graph obtained from G by giving weights to its edges using the positive weight function w. The

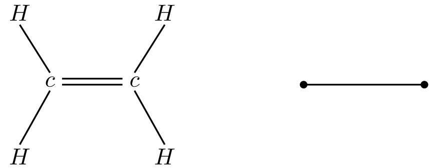

Fig. 1. An ethylene molecule and its Hückel graph.