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The coefficients of the reduced Bartholdi zeta function

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ABSTRACT

In this paper, we establish a new zeta function based on the Bartholdi zeta function for an undirected graph G called the reduced Bartholdi zeta function. We study the relation between its coefficients and the structure of the graph, and demonstrate that the coefficients count the star subgraphs in the symmetric digraph $\mathcal{D}(G)$. Moreover, we investigate the properties of semi-principle minors extracted from the adjacency matrix of the oriented line graph of G. We also present a general formula for calculating all the coefficients of the reduced Bartholdi zeta function.

1. Introduction

The Ihara zeta function was first introduced by Ihara [6] in 1966 for finite k-regular graphs, in the context of discrete groups of the p-adic zeta functions and it was shown that the Ihara zeta function can be represented in the form of the reciprocal of a polynomial. In 1989, Hashimoto [4] deduced multi-variable zeta functions for bi-regular bipartite

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graphs. Also, a generalization of the determinant expression of the Ihara zeta function to all finite irregular graphs by its adjacency matrix was performed by Bass [2]. Further, Horton [5] demonstrated the relation between the girth of graphs and the Ihara zeta function. Stark and Terras [14,17] argued about various types of the zeta functions for any graph and presented a comprehensive overview of the zeta functions. In 2008, an investigation about the coefficients of the Ihara zeta functions and their associated cyclical structure in graph has been followed by Scott and Storm [13].

The two-variable zeta function of a graph was first demonstrated by Bartholdi [1]. He obtained a determinant expression for the two-variable zeta function, called the Bartholdi zeta function and gave a formula for concluding the number of bumps on the paths in a graph. Mizuno and Sato [8,9] defined a new zeta function for a digraph by using a determinant expression of the zeta function and also expressed its weighted Bartholdi zeta function. For generalizing the Bartholdi zeta function, Sato [11], represented a three-variable zeta function of a bipartite graph and presented its determinant expression. Moreover, in [12], he presented an $(n+1)$-variable Bartholdi L-function of graph G. By the weighted scattering matrix of G, Oren [10] stated a new version of the Bartholdi zeta function.

There is a substantial research literature on studying the properties of the Ihara zeta function [4,5,7], in distinguishing co-spectral graphs [15] and on developing the various types of the Ihara zeta function [3,16].

However, what is still lacking is an explicit description of the relation between the Bartholdi zeta function and the structure of a graph. Furthermore, despite the Ihara zeta function can be computed in polynomial time, the Bartholdi zeta function is not easy to compute, since it is expressible as a two-variable determinant. Note that all types of zeta functions are defined on md2 graphs in which each vertex has at least degree 2.

In this paper, we define a new Bartholdi zeta function of a graph called the reduced Bartholdi zeta function. We investigate about the coefficients of the reduced Bartholdi zeta function and demonstrate that these coefficients can count the special complete bipartite structures as stars in the graph.

The reduced Bartholdi zeta function focuses on the variable related to the number of bumps on paths in a graph. In other words, we intend to concentrate on the exclusive variable of the Bartholdi zeta function. To reach this aim, we ignore the other variable which is common with the Ihara zeta function. Moreover, the other advantage of omitting this variable lies in the fact that the determinant expression of the reduced Bartholdi zeta function can be computed in polynomial time.

The rest of the paper is organized as follows. In the next section, we establish the definition of the Bartholdi zeta function and present some results and preliminaries which are necessary for the remainder of the paper. In Section 3, the *semi-principle minors* of the reduced Bartholdi zeta function are defined and their properties are studied. Section 4 presents the relation between the structure of graphs and the first five coefficients of the reduced Bartholdi zeta function. Finally, in Section 5, we achieve a general formula for calculating all the coefficients of the reduced Bartholdi zeta function.