Linear Algebra and its Applications 504 (2016) 1–13

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

A structure theory for graphs with fixed smallest eigenvalue

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ARTICLE INFO

Article history:
Received 20 April 2015
Accepted 25 March 2016
Available online 6 April 2016
Submitted by R. Brualdi

Dedicated to Alan J. Hoffman on the occasion of his 91st birthday

MSC:
05C50
05C75
05C62

Keywords:
Smallest eigenvalue
Structure theory
Hoffman graph
Quasi-clique

ABSTRACT

In this paper, we will give a structure theory for graphs with fixed smallest eigenvalue. In order to do this, the concept of Hoffman graph (as introduced by Woo and Neumaier) is used. Our main result states that for fixed positive integer λ and any graph G with smallest eigenvalue at least $-\lambda$, there exist dense induced subgraphs $Q_1, \ldots, Q_c$ in G such that each vertex lies in at most λ Q_i's and almost all edges of G lie in at least one of the Q_i's.

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http://dx.doi.org/10.1016/j.laa.2016.03.044

1. Introduction

For graph theoretic notations, we refer to [2]. All graphs in this paper are undirected and simple. In this paper, we will give a structure theory for graphs with fixed smallest eigenvalue. We will use the concept of Hoffman graph (as introduced by Woo and Neumaier [7]). First, we will introduce a result of Hoffman.

Let t be a positive integer and let $\tilde{K}_{2t}$ be the graph on $2t+1$ vertices consisting of a complete graph K_{2t} and a vertex ∞ which is adjacent to exactly t vertices of the K_{2t}. It is easy to see that the smallest eigenvalue of $\tilde{K}_{2t}$ goes to $-\infty$ if t goes to ∞. The smallest eigenvalue of a t-claw $K_{1,t}$ equals $-\sqrt{t}$ (where t is a positive integer), and hence it will go to $-\infty$ if t goes to ∞. So if the smallest eigenvalue of a graph G is at least a fixed real number λ, then there exists a positive integer $t = t(\lambda)$ such that G contains neither a $\tilde{K}_{2t}$ nor a t-claw $K_{1,t}$ as an induced subgraph. Hoffman [3] showed that the opposite is also true:

Theorem 1.1. *Let G be a graph with smallest eigenvalue $\lambda_{\min}(G)$. Then the following holds.*

(i) *For any real number $\lambda \leq -1$, there exists a positive integer $t = t(\lambda)$ such that if $\lambda_{\min}(G) \geq \lambda$, then G contains neither a $\tilde{K}_{2t}$ nor a t-claw $K_{1,t}$ as an induced subgraph.*
(ii) *For any positive integer t, there exists a non-positive real number $\lambda = \lambda(t)$ such that if G contains neither a $\tilde{K}_{2t}$ nor a t-claw $K_{1,t}$ as an induced subgraph, then $\lambda_{\min}(G) \geq \lambda$.*

In order to prove Theorem 1.1(ii), Hoffman showed that if G contains neither a $\tilde{K}_{2t}$ nor a t-claw $K_{1,t}$ as an induced subgraph, then there exists a highly structured graph H with the same vertex set as G and with large distinguished cliques that is close to G (see, for example, [1, Theorem 7.3.1] for a precise description of the graph H). In this paper, we will give a proof by using the concept of Hoffman graphs.

The main focus of this paper is to obtain a structure theory for graphs with fixed smallest eigenvalue. Our main theorem is:

Theorem 1.2. *Let $\lambda \leq -1$ be a real number. Then there exists a positive integer d_λ such that if G is a graph with smallest eigenvalue $\lambda_{\min}(G) \geq \lambda$ and its minimal valency $\delta(G) \geq d_\lambda$, then for some integer c, there exist induced subgraphs $Q_1, Q_2, \ldots, Q_c$ satisfying the following conditions:*

(i) *Each vertex x of G lies in at least one and at most $-\lambda$ Q_i's;*
(ii) *For each i, the complement of Q_i has maximal valency at most λ^2;*
(iii) *For $i \neq j$, the intersection $V(Q_i) \cap V(Q_j)$ contains at most $-\lambda - 1$ vertices;*
(iv) *The graph $G' := (V(G), E(G) \setminus \bigcup_{i=1}^{c} E(Q_i))$ has maximal valency at most d_λ.*