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Sharp spectral bounds of several graph parameters using eigenvector norms

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ABSTRACT

We investigate the role of the 1- and ∞-norms of eigenvectors in spectral graph theory. In particular, we produce several randomized algorithms which show that various graph-theoretic parameters can be tightly bounded by the eigenvalues as well as norms of the corresponding eigenvectors. Further, in some cases, these inequalities can determine the parameters exactly. Our results include: a spectral bound for the densest subgraph problem, an adapted “converse” to the Expander Mixing Lemma, and an adapted spectral approach to the maximum cut problem.

1. Introduction

In this article, we will derive spectral bounds for three combinatorial problems: the densest subgraph problem, discrepancy, and the maximum cut problem.

Our contribution is to provide several spectral bounds for these problems that consider the eigenvector norms in addition to the eigenvalues. In some cases, this may provide an improvement over existing bounds. There are to common threads among these results.

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First, the main techniques we use are probabilistic. In particular, in order to prove our results, we produce a randomized algorithm based on the eigenvalues and eigenvectors to generate an instance (e.g., a set, a pair of sets, or a cut) which achieves the given bound in expectation. The second common element is the notion that eigenvector norms can elegantly bound these combinatorial parameters. In fact, a recurring theme will be to provide "sandwich" variations of the norm-equivalence inequalities.

The densest subgraph of a graph is the subgraph with the largest average degree. In many cases, the densest subgraph is in fact the whole graph itself. There are polynomial-time algorithms to find the optimal subgraph [11]. However, a more interesting problem arises when the size of the desired subgraph is fixed or constrained; in which case, finding the optimal subgraph is NP-hard [9]. In Section 3, we relate the maximum subgraph density, ρ_G, to the maximum eigenvalue by proving the following (Theorem 1):

$$\frac{\lambda_{\max}}{\|\mathbf{v}\|_1\|\mathbf{v}\|_\infty} \leq \rho_G \leq \lambda_{\max} \tag{1}$$

where $\mathbf{v}$ is a unit eigenvector corresponding to $\lambda_{\max}$.

Next, in Section 4, we investigate the relationship between the discrepancy of the graph and the second largest singular value, $\sigma_2 = \max[\lambda_2, -\lambda_{\min}]$. Loosely speaking, the discrepancy describes how "random-like" a graph is by comparing the portion of vertices within two sets to the number of edges between them. Specifically, the discrepancy of a graph on n vertices with edge density $\rho := \frac{2|E|}{n(n-1)}$ (or, for directed graphs, $= \frac{|E|}{n(n-1)}$) is defined as the least constant, β, such that for all sets of vertices S, T:

$$|E(S,T) - \rho|S||T|| \leq \beta\sqrt{|S||T|}.$$

The Expander Mixing Lemma [1] and its celebrated "converse" by Bilu and Linial, for regular graphs [4], and Bollobás and Nikiforov, more generally [5], show that the discrepancy of an undirected graph is tightly controlled by σ_2 to within a logarithmic factor. For regular graphs,

$$\beta \leq \sigma_2 \leq k\beta\left(1 + \log\frac{d}{\beta}\right) \tag{2}$$

where the left-hand side is the standard Expander Mixing Lemma and the right-hand side is the "converse." Moreover, this logarithmic factor is necessary [4,5]. We will derive an alternative "converse" Expander Mixing Lemma by replacing this logarithmic factor with a factor determined by the norms of the corresponding eigenvector. We prove (Theorem 2):

$$\frac{\sigma_2}{\|\mathbf{v}\|_1\|\mathbf{v}\|_\infty} \leq \beta \leq \sigma_2 \tag{3}$$