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Large spaces of bounded rank matrices revisited

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Let n, p, r be positive integers with $n \geq p \geq r$. A rank-$\overline{r}$ subset of n by p matrices (with entries in a field) is a subset in which every matrix has rank less than or equal to r. A classical theorem of Flanders states that the dimension of a rank-$\overline{r}$ linear subspace must be less than or equal to nr, and it characterizes the spaces with the critical dimension nr. Linear subspaces with dimension close to the critical one were later studied by Atkinson, Lloyd and Beasley over fields with large cardinality; their results were recently extended to all fields [18].

Using a new method, we obtain a classification of rank-$\overline{r}$ affine subspaces with large dimension, over all fields. This classification is then used to double the range of (large) dimensions for which the structure of rank-$\overline{r}$ linear subspaces is known for all fields.

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1. Introduction

1.1. The context

Throughout the text, we fix an arbitrary field and denote it by $\mathbb{K}$. Given non-negative integers n and p, we denote by $\mathrm{M}_{n,p}(\mathbb{K})$ the set of all matrices with n rows, p columns and

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entries in $\mathbb{K}$. The rank of a matrix M is denoted by $\operatorname{rk}(M)$, while the transpose of M is denoted by M^T. We set $\mathrm{M}_n(\mathbb{K}) := \mathrm{M}_{n,n}(\mathbb{K})$ and we denote by $\mathrm{GL}_n(\mathbb{K})$ the group of units of the ring $\mathrm{M}_n(\mathbb{K})$. We denote by $T_n^+(\mathbb{K})$ the subspace of all upper-triangular matrices of $\mathrm{M}_n(\mathbb{K})$.

Subsets $\mathcal{V}$ and $\mathcal{W}$ of $\mathrm{M}_{n,p}(\mathbb{K})$ are called **equivalent** when there exist invertible matrices $P \in \mathrm{GL}_n(\mathbb{K})$ and $Q \in \mathrm{GL}_p(\mathbb{K})$ such that $\mathcal{V} = P\,\mathcal{W}\,Q$ (in other words, $\mathcal{V}$ and $\mathcal{W}$ represent, in a different choice of bases, the same set of linear transformations from a p-dimensional vector space to an n-dimensional vector space).

The **upper-rank** of a non-empty subset $\mathcal{V}$ of $\mathrm{M}_{n,p}(\mathbb{K})$, denoted by $\operatorname{urk}\mathcal{V}$, is defined as the maximal rank among the matrices of $\mathcal{V}$. Given a non-negative integer $r \in [\![0, \min(n,p)]\!]$, a **rank-$\overline{r}$** subset of $\mathrm{M}_{n,p}(\mathbb{K})$ is a subset $\mathcal{V}$ such that $\operatorname{urk}\mathcal{V} \leq r$. A classical example of such subsets is the so-called compression spaces: given integers $s \in [\![0,n]\!]$ and $t \in [\![0,p]\!]$, one defines

$$\mathcal{R}(s,t) := \left\{ \begin{bmatrix} A & C \\ B & [0]_{(n-s)\times(p-t)} \end{bmatrix} \mid A \in \mathrm{M}_{s,t}(\mathbb{K}),\ B \in \mathrm{M}_{n-s,t}(\mathbb{K}),\ C \in \mathrm{M}_{s,p-t}(\mathbb{K}) \right\}.$$

If $s+t \leq \min(n,p)$, then one checks that $\mathcal{R}(s,t)$ is a rank-$\overline{s+t}$ linear subspace of $\mathrm{M}_{n,p}(\mathbb{K})$ with dimension $nt+s(p-t)$. A rank-$\overline{r}$ **compression space** is a matrix subspace of $\mathrm{M}_{n,p}(\mathbb{K})$ that is equivalent to $\mathcal{R}(s,t)$ for some non-negative integers s and t such that $s+t=r$ and $r \leq \min(n,p)$. A subset $\mathcal{V}$ of $\mathrm{M}_{n,p}(\mathbb{K})$ is called **r-decomposable** when it is included in a rank-$\overline{r}$ compression space: in terms of operators, this means that there are non-negative integers s and t such that $s+t=r$, a $(p-t)$-dimensional linear subspace G of $\mathbb{K}^p$ and an s-dimensional linear subspace H of $\mathbb{K}^n$ such that every matrix of $\mathcal{V}$ maps G into H.

Of course, every subset of a rank-$\overline{r}$ subset is also a rank-$\overline{r}$ subset, and every subset that is equivalent to a rank-$\overline{r}$ subset is a rank-$\overline{r}$ subset. Thus, in trying to understand the structure of rank-$\overline{r}$ subspaces, one should focus on the equivalence classes of the *maximal* ones. It is easy to prove that if $s+t=r$, then $\mathcal{R}(s,t)$ is a maximal rank-$\overline{r}$ affine subspace and the compression spaces $\mathcal{R}(i,r-i)$, for $i \in [\![0,r]\!]$, are pairwise inequivalent. However, not every maximal rank-$\overline{r}$ linear subspace is a compression space. A classical example is the one where n is odd and greater than 1, and where $p=n$ and $r=n-1$: then, the space $\mathrm{A}_n(\mathbb{K})$ of all *alternating* n by n matrices is a maximal rank-$\overline{n-1}$ linear subspace of $\mathrm{M}_n(\mathbb{K})$ (see [7] for fields with more than 2 elements, and [13] for fields with two elements); yet it is easily checked that it is not $(n-1)$-decomposable.

Classifying the maximal rank-$\overline{r}$ subspaces is generally viewed as an intractable problem. To get meaningful results, one needs to restrict the scope of the research. One such possible restriction is to focus on small values of r only: solutions to this problem are known for $r \leq 3$ except for very small fields (see [1]). For general values of r, another approach is to focus on the so-called primitive subspaces [3,6]; this approach is generally well-suited to classify rank-$\overline{r}$ spaces for small values of r, but it also has surprising connections with the topic of large spaces of nilpotent matrices [10]. Finally, the most classical approach, which dates back to works of Dieudonné [5] and Flanders [8], consists

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