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Bounds for eigenvalues of matrix polynomials with applications to scalar polynomials

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ABSTRACT

We first generalize to complex matrix polynomials an improvement of an upper bound by Cauchy on the zeros of complex scalar polynomials. The bound requires the unique positive root of a real scalar polynomial of the same degree as the given complex scalar or matrix polynomial. We then create a recursive procedure to represent a matrix polynomial by another matrix polynomial of larger size, but of lower degree. We apply this procedure to scalar polynomials, and then apply the generalized improved Cauchy bound to their matrix polynomial representation, often obtaining a better bound by solving a real scalar polynomial equation of significantly reduced degree.

1. Introduction

We generalize to matrix polynomials a result for scalar polynomials that, as far as we know, was derived for the first time in Theorem 8.3.1 of [12]. It provides at least as good an upper bound on the moduli of the zeros of a polynomial as that given by a

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well-known result by Cauchy ([3], [10, Th. (27,1), p. 122 and Exercise 1, p. 126]), which states that all the zeros of the polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ with complex coefficients and $a_n \neq 0$, lie in $|z| \leq r$, where r is the unique positive solution of

$$|a_n| x^n - |a_{n-1}| x^{n-1} - \cdots - |a_1| x - |a_0| = 0 .$$

We call r the *Cauchy radius* of p. A similar upper bound for the moduli of the eigenvalues of a matrix polynomial was derived in [2], [8], and [11]. It states that all the eigenvalues of the matrix polynomial $P(z) = A_n z^n + A_{n-1} z^{n-1} + \cdots + A_1 z + A_0$ with complex $m \times m$ coefficient matrices and A_n nonsingular, lie in $|z| \leq s$, where s is the unique positive solution of

$$\|A_n^{-1}\|^{-1} x^n - ||A_{n-1}|| x^{n-1} - \cdots - ||A_1|| x - ||A_0|| = 0 .$$

The matrix norms are assumed to be subordinate (induced by a vector norm). Analogously to the scalar case, we call s the *Cauchy radius* of P. Matrix polynomials appear in generalized eigenvalue problems where a nonzero complex vector v and a complex number z are sought such that $P(z)v = 0$. If A_n is singular then there are infinite eigenvalues and if A_0 is singular then zero is an eigenvalue. There are nm eigenvalues, including possibly infinite ones. The finite eigenvalues of the matrix polynomial P are the zeros of the (scalar) polynomial $\det P(z)$. A good starting point to gain an idea of the importance of such problems in engineering is [14].

There are many methods available for computing the positive solution of the real scalar equations mentioned here and we will not dwell on them as this matter is irrelevant to our results.

Our goal is to derive a matrix version of the main statement of Theorem 8.3.1 in [12] to improve the matrix version of Cauchy's result, just as Theorem 8.3.1 improves the classical (scalar) version of Cauchy's result. The resulting improvement is more important in the matrix case, since computing eigenvalues of matrix polynomials is significantly more difficult than computing polynomial zeros. Among others, bounds on such eigenvalues are useful in their computation by iterative methods [13] and when computing pseudospectra [7,15]. We first illustrate our result by computing bounds on the eigenvalues of several matrix polynomials, some occurring in engineering applications and some that were randomly generated, before applying it – paradoxically – to scalar polynomials. We show how a scalar polynomial can be represented by a matrix polynomial of lower degree to which the generalized improved Cauchy result can be applied to obtain a bound on the moduli of its zeros that in many cases is comparable to or better than its Cauchy radius, while only requiring the positive root of a real scalar polynomial of significantly reduced degree.

The paper is organized as follows. In Section 2 we derive the generalization of Theorem 8.3.1 in [12] that we then apply in Section 3 to several examples of matrix