Linear Algebra and its Applications 504 (2016) 248–271

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

The Voronoi inverse mapping

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ARTICLE INFO

Article history:
Received 25 October 2014
Accepted 2 April 2016
Available online 14 April 2016
Submitted by R. Brualdi

MSC:
15A39
51M20
52C22

Keywords:
Voronoi cells
Linear inequality systems
Inverse problem
Voronoi inverse mapping

ABSTRACT

Given an arbitrary set T in the Euclidean space whose elements are called sites, and a particular site s, the Voronoi cell of s, denoted by $V_T(s)$, consists of all points closer to s than to any other site. The Voronoi mapping of s, denoted by ψ_s, associates to each set $T \ni s$ the Voronoi cell $V_T(s)$ of s w.r.t. T. These Voronoi cells are solution sets of linear inequality systems, so they are closed convex sets. In this paper we study the Voronoi inverse problem consisting in computing, for a given closed convex set $F \ni s$, the family of sets $T \ni s$ such that $\psi_s(T) = F$. More in detail, the paper analyzes relationships between the elements of this family, $\psi_s^{-1}(F)$, and the linear representations of F, provides explicit formulas for maximal and minimal elements of $\psi_s^{-1}(F)$, and studies the closure operator that assigns, to each closed set T containing s, the largest element of $\psi_s^{-1}(F)$, where $F = V_T(s)$.

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http://dx.doi.org/10.1016/j.laa.2016.04.004
0024-3795/

1. Introduction

Let $T \subset \mathbb{R}^n$, with $n \geq 1$, be a set whose elements are called *Voronoi sites*. The *Voronoi cell of* $s \in T$, denoted by $V_T(s)$, consists of all points closer to s than to any other site, i.e.,

$$V_T(s) := \{x \in \mathbb{R}^n : d(x,s) \leq d(x,t),\ t \in T\},$$

where d denotes the Euclidean distance on $\mathbb{R}^n$. In other terms, denoting by $P_T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ the metric projection on T, i.e., $P_T(x) := \{t \in T : d(x,t) = d(x,T)\}$, then $V_T(s) = P_T^{-1}(s)$. The *Voronoi diagram* of T is $\operatorname{Vor}(T) := \{V_T(t), t \in T\}$. This family of closed convex sets is a tessellation of $\mathbb{R}^n$ if and only if T is closed [6, Proposition 1], which is the general assumption of this paper. When T is finite, $\operatorname{Vor}(T)$ is formed by full dimensional convex polyhedral sets.

Voronoi cells of finite sets of sites in two, three, and $n > 3$ dimensions were introduced by Descartes, Dirichlet, and Voronoi, in 1644, 1850, and 1908, respectively. Voronoi cells of finite sets are widely applied in computational geometry, operations research, data compression, economics, marketing, etc. (see, e.g., [1] and [9]). In 1934 Delaunay [2, General Lemma] introduced Voronoi cells of discrete sets of sites (i.e., sets without accumulation points) with applications in crystallography. Recently, Voigt and Weis, [12,13], obtained results on Voronoi cells of discrete sets exploiting the fact that $V_T(s)$ is the solution set of the linear system

$$\upsilon_T(s) := \left\{(t-s)' x \leq \tfrac{\|t\|^2 - \|s\|^2}{2}, t \in T \setminus \{s\}\right\}, \tag{1}$$

where t' and $\|t\|$ denote the transpose of t and the Euclidean norm of t, respectively. A similar approach was used in [6] in order to obtain, in a systematic way, geometric information on $V_T(s)$ in terms of the data (T and s) for different types of infinite sets of sites such as curves, closed convex sets, etc. The latter work shows that Voronoi cells of discrete sets are useful in location decision making while Voronoi cells of non-discrete sets can be used to assign clients for services delivered along curves (such as rivers or highways) or for services delivered at a finite set of uncertain sites (robust approach). Let us mention here that Voronoi cells of infinite sets are of potential use for delimiting international waters among neighbor countries by assuming that each point should be assigned to the closest country. Then each country should get the union of the Voronoi cells corresponding to the points of the national shore together with a proportional part of the Voronoi cells of the border points. For instance, in the simple case of a compact convex island T shared by two countries occupying non-overlapping territories $A, B \subset T$, the first country would get

$$\bigcup \{V_T(s) : s \in \operatorname{bd} T \cap (A \setminus B)\}$$

together with one half of