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Standard polynomials and matrices with superinvolutions

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A B S T R A C T

Let $M_n(F)$ be the algebra of $n \times n$ matrices over a field F of characteristic zero. The superinvolutions $*$ on $M_n(F)$ were classified by Racine in [12]. They are of two types, the transpose and the orthosymplectic superinvolution. This paper is devoted to the study of $*$-polynomial identities satisfied by $M_n(F)$. The goal is twofold. On one hand, we determine the minimal degree of a standard polynomial vanishing on suitable subsets of symmetric or skew-symmetric matrices for both types of superinvolutions. On the other, in case of $M_2(F)$, we find generators of the ideal of $*$-identities and we compute the corresponding sequences of cocharacters and codimensions.

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1. Introduction

Let $St_r(x_1, \ldots, x_r) = \sum_{\sigma \in S_r} (\operatorname{sgn} \sigma) x_{\sigma(1)} \cdots x_{\sigma(r)}$ be the standard polynomial of degree r in the non-commutative variables $x_1, \ldots, x_r$. A celebrated theorem of Amitsur and Levitzki states that St_{2n} is an identity for the algebra of $n \times n$ matrices over a commutative ring. The original proof (see [2]) was highly combinatorial and based on the properties of the matrix units. Several different proofs were given afterwards (see for instance [13,15,18,19]).

Among them, we recall the proof given by Konstant in [10] who reproved the Amitsur–Levitzki theorem by showing that it is equivalent to a theorem in Lie cohomology. Moreover, he showed the power of his method by proving that $St_{2n-2}(K_1, \ldots, K_{2n-2}) = 0$, for all n even, where K_i, $i = 1, \ldots, 2n-2$, are $n \times n$ skew-symmetric matrices with respect to the transpose involution. In 1974, Rowen (see [16]) reproved Kostant's theorem through a graph-theoretical approach and was able to extend this result to the case n odd (see also [9]). In this way a general question whether the Amitsur–Levitzki theorem could be improved by considering only certain kind of matrices was posed. If the field F has characteristic zero, it is not hard to prove that St_{2n} is, up to a scalar, the only identity of minimal degree of $M_n(F)$. Hence a general question whether a similar phenomenon appears when dealing with symmetric or skew-symmetric matrices was considered.

When Procesi and Razmyslov in [11,13] showed that the Amitsur–Levitzki theorem followed formally from the Hamilton–Cayley polynomial, renewed interest was aroused to prove Kostant's results by this method. Considering this approach, Rowen in [17] presented a simple proof of Kostant's theorem and solved the analogous question for the symplectic involution s. In particular he showed that $St_{2n-2}(S_1, \ldots, S_{2n-2}) = 0$, where S_i, $i = 1, \ldots, 2n-2$, are $n \times n$ symmetric matrices with respect to s.

The purpose of this paper is to prove similar results in the setting of matrix algebras with superinvolution.

If we write $n = k + h$, then $A = M_n(F)$ becomes a superalgebra $A = A_0 \oplus A_1$, where

$$A_0 = \left\{ \begin{pmatrix} X & 0 \\ 0 & T \end{pmatrix} \mid X \in M_k(F),\ T \in M_h(F) \right\},$$

$$A_1 = \left\{ \begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix} \mid Y \in M_{k\times h}(F),\ Z \in M_{h\times k}(F) \right\}.$$

Such superalgebra is denoted by $M_{k,h}(F)$. If F is a field of characteristic zero, it is well known that any non trivial $\mathbb{Z}_2$-grading on $M_n(F)$ is isomorphic to $M_{k,h}(F)$ for some k and h.

In [12], Racine described all types of superinvolutions on $M_{k,h}(F)$ by showing that, up to isomorphism, we have only the orthosymplectic and the transpose superinvolution. We shall give the definitions of such superinvolutions in the next section.

In the first part of this paper, we find the minimal degree for which the standard polynomial vanishes when evaluated in homogeneous symmetric or skew-symmetric matrices