Linear Algebra and its Applications 504 (2016) 292–308

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Numerical radius of Hadamard product of matrices

Hwa-Long Gau [a,*,1], Pei Yuan Wu [b]

[a] Department of Mathematics, National Central University, Chungli 32001, Taiwan
[b] Department of Applied Mathematics, National Chiao Tung University, Hsinchu 30010, Taiwan

ARTICLE INFO

Article history:
Received 12 January 2016
Accepted 10 April 2016
Available online 18 April 2016
Submitted by C.-K. Li

MSC:
15A60
15B48

Keywords:
Numerical radius
Hadamrad product
Normal matrix
Positive semidefinite matrix

ABSTRACT

It is known that the numerical radius of the Hadamard product $A \circ B$ of two n-by-n matrices A and B is related to those of A and B by (a) $w(A \circ B) \leq 2w(A)w(B)$, (b) $w(A \circ B) \leq w(A)w(B)$ if one of A and B is normal, and (c) $w(A \circ B) \leq (\max_i a_{ii})w(B)$ if $A = [a_{ij}]_{i,j=1}^n$ is positive semidefinite. In this paper, we give complete characterizations of A and B for which the equality is attained. The matrices involved can be considered as elaborate generalizations of the equality-attaining $A = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$ for (a), $A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$ ($|a_1| \geq |a_2|$) and $B = \begin{bmatrix} w(B) & * \\ * & * \end{bmatrix}$ for (b), and $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \geq 0$ ($a_1 \geq a_4$) and $B = \begin{bmatrix} w(B) & * \\ * & * \end{bmatrix}$ for (c).

1. Introduction

If $A = [a_{ij}]_{i,j=1}^n$ and $B = [b_{ij}]_{i,j=1}^n$ are n-by-n complex matrices, then their *Hadamard product* $A \circ B$ is the matrix $[a_{ij}b_{ij}]_{i,j=1}^n$. The *numerical range* of A is the subset $W(A) = \{\langle Ax, x\rangle : x \in \mathbb{C}^n, \|x\| = 1\}$ of the complex plane, where $\langle \cdot, \cdot\rangle$ and $\|\cdot\|$ denote the standard

* Corresponding author.
E-mail addresses: hlgau@math.ncu.edu.tw (H.-L. Gau), pywu@math.nctu.edu.tw (P.Y. Wu).
[1] Research supported by the Ministry of Science and Technology of the Republic of China under MOST 104-2115-M-008-006.

http://dx.doi.org/10.1016/j.laa.2016.04.013
0024-3795/

inner product and norm of vectors in $\mathbb{C}^n$, respectively, and the *numerical radius* of A is $w(A) = \max\{|z| : z \in W(A)\}$. It is known that the numerical radii of A, B and $A \circ B$ are related via the inequalities: (a) $w(A \circ B) \leq 2w(A)w(B)$, (b) $w(A \circ B) \leq w(A)w(B)$ if A is normal, and (c) $w(A \circ B) \leq (\max_i a_{ii})w(B)$ if A is positive semidefinite. Here (a) is a consequence of the following series of inequalities:

$$w(A \circ B) \leq w(A \otimes B) \leq \min\{\|A\|w(B), \|B\|w(A)\} \leq 2w(A)w(B), \tag{1.1}$$

where the first inequality follows from the fact that $A \circ B$ is a principal submatrix of the tensor product $A \otimes B$, and the second and third from [3, Proposition 1.1] and [5, p. 44, Problem 23 (g)], respectively. On the other hand, (b) is by [5, Corollary 4.2.17] and (c) by [1, Corollary 4]. In this paper, we completely characterize those A's and B's for which the equality is attained in the above cases. One example for the equality in (a) is $A = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$, for (b) is $A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$ ($|a_1| \geq |a_2|$) and $B = \begin{bmatrix} w(B) & * \\ * & * \end{bmatrix}$, and for (c) $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \geq 0$ ($a_{11} \geq a_{22}$) and $B = \begin{bmatrix} w(B) & * \\ * & * \end{bmatrix}$. Our characterizations can be considered as far-fetching generalizations of such examples. These will be taken up in the subsequent sections.

In Section 2 below, we consider when the equality $w(A \circ B) = 2w(A)w(B)$ holds. A complete characterization is given in Theorem 2.1. In case $n = 2$, this can be simplified to a manageable form in Corollary 2.2: two nonzero 2-by-2 matrices A and B are such that $w(A \circ B) = 2w(A)w(B)$ if and only if $A = U^* \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} U$ and $B = V^* \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} V$ for some unitary matrices $U = [u_{ij}]_{i,j=1}^2$ and $V = [v_{ij}]_{i,j=1}^2$ with $|u_{ij}| = |v_{ij}|$ for all i and j. Section 3 deals with $w(A \circ B) = w(A)w(B)$ for normal A. The main result is Theorem 3.2. Proposition 3.1 gives the special case when A is already diagonal: for the n-by-n matrices $A = \mathrm{diag}(a_1, \ldots, a_n)$ with $|a_1| = \cdots = |a_k| > |a_{k+1}|, \ldots, |a_n|$ ($1 \leq k \leq n$) and $B = [b_{ij}]_{i,j=1}^n$, the equality $w(A \circ B) = w(A)w(B)$ holds if and only if $w(B) = |b_{jj}|$ for some j, $1 \leq j \leq k$. In Section 4, we consider the equality $w(A \circ B) = (\max_i a_{ii})w(B)$ for the matrices $A = [a_{ij}]_{i,j=1}^n \geq 0$ and $B = [b_{ij}]_{i,j=1}^n$. We start with a direct proof of $w(A \circ B) \leq (\max_i a_{ii})w(B)$ in Proposition 4.1. The inequality was first obtained by Ando and Okubo [1, Corollary 4]. The equality case is discussed in Theorem 4.2, a particular case of which, when A is positive definite, says that the equality holds if and only if $w(B) = |b_{jj}|$ for some j with $a_{jj} = \max_i a_{ii}$ (cf. Corollary 4.3). We conclude with the result on a condition for a positive semidefinite A to satisfy $w(A \circ B) = (\max_i a_{ii})w(B)$ for all matrices B of size n (cf. Theorem 4.6).

Admittedly, the conditions in our characterizations are in general difficult to verify. These make their special cases more interesting. The difficulties originate from the fact that the Hadamard product is basically a matrix operation instead of an operator one. This means that even for matrices A and B which are simultaneously unitarily similar to C and D, their Hadamard products may not be unitarily similar. One such example is $A = B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $C = D = (1/2)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, in which case A, B, C and D are all unitarily similar to each other, but $A \circ B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $C \circ D = (1/4)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ are not.