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Taylor's theorem for matrix functions with applications to condition number estimation ☆

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We derive an explicit formula for the remainder term of a Taylor polynomial of a matrix function. This formula generalizes a known result for the remainder of the Taylor polynomial for an analytic function of a complex scalar. We investigate some consequences of this result, which culminate in new upper bounds for the level-1 and level-2 condition numbers of a matrix function in terms of the pseudospectrum of the matrix. Numerical experiments show that, although the bounds can be pessimistic, they can be computed much faster than the standard methods. This makes the upper bounds ideal for a quick estimation of the condition number whilst a more accurate (and expensive) method can be used if further accuracy is required. They are also easily applicable to more complicated matrix functions for which no specialized condition number estimators are currently available.

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1. Introduction

Taylor's theorem is a standard result in elementary calculus (see e.g. [17]). If $f : \mathbb{R} \to \mathbb{R}$ is k times continuously differentiable at $a \in \mathbb{R}$, then the theorem states that there exists $R_k : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = \sum_{j=0}^{k} \frac{f^{(j)}(a)}{j!}(x-a)^j + R_k(x)$$

and $R_k(x) = o(|x-a|^k)$ as $x \to a$. Depending on any additional assumptions on f, various precise formulae for the remainder term $R_k(x)$ are available. For example, if f is $k+1$ times continuously differentiable on the closed interval between a and x, then

$$R_k(x) = \frac{f^{(k+1)}(c)}{(k+1)!}(x-a)^{k+1} \tag{1}$$

for some c between a and x. This is known as the Lagrange form of the remainder. Alternative expressions, such as the Cauchy form or the integral form for the remainder are well known [17].

Taylor's theorem generalizes to analytic functions in the complex plane: the remainder must now be expressed in terms of a contour integral. If $f(z)$ is complex analytic in an open subset $\mathcal{D} \subset \mathbb{C}$ of the complex plane, the kth-degree Taylor polynomial of f at $a \in \mathcal{D}$ satisfies

$$f(z) = \sum_{j=0}^{k} \frac{f^{(k)}(a)}{k!}(z-a)^j + R_k(z),$$

where

$$R_k(z) = \frac{(z-a)^{k+1}}{2\pi i} \int_{\Gamma} \frac{f(w)dw}{(w-a)^{k+1}(w-z)}, \tag{2}$$

and Γ is a circle, centred at a, such that $\Gamma \subset \mathcal{D}$. See [1, Chap. 5, Sec. 1.2] for a proof of this result.

The first goal of this paper is to generalize (2) to matrices, thereby providing an explicit expression for the remainder term for the kth-degree Taylor polynomial of a matrix function. Note that it will not be possible to obtain an expression similar to (1) because its derivation relies on the mean value theorem which does not have an exact analogue for matrix-valued functions. Our second goal is to investigate applications of this result in bounding the derivatives and condition numbers of matrix functions via pseudospectra.

Convergence results for Taylor polynomials of matrix functions have been known since the work of Hensel [8], Turnbull [20], and Weyr [21] (see [11, Thm. 4.7] for a more recent