Linear Algebra and its Applications 504 (2016) 581–589

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Topological classification of sesquilinear forms: Reduction to the nonsingular case

Carlos M. da Fonseca [a,*], Tetiana Rybalkina [b], Vladimir V. Sergeichuk [b]

[a] Department of Mathematics, Kuwait University, Safat 13060, Kuwait
[b] Institute of Mathematics, Tereshchenkivska 3, Kiev, Ukraine

A R T I C L E I N F O

Article history:
Received 25 February 2016
Accepted 13 April 2016
Available online 26 April 2016
Submitted by R. Brualdi

MSC:
15A21
15B33
37J40

Keywords:
Topological equivalence
Regularizing decomposition
Bilinear and sesquilinear forms

A B S T R A C T

Two sesquilinear forms $\Phi : \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}$ and $\Psi : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ are called topologically equivalent if there exists a homeomorphism $\varphi : \mathbb{C}^m \to \mathbb{C}^n$ (i.e., a continuous bijection whose inverse is also a continuous bijection) such that $\Phi(x, y) = \Psi(\varphi(x), \varphi(y))$ for all $x, y \in \mathbb{C}^m$. R.A. Horn and V.V. Sergeichuk in 2006 constructed a regularizing decomposition of a square complex matrix A; that is, a direct sum $SAS^* = R \oplus J_{n_1} \oplus \cdots \oplus J_{n_p}$, in which S and R are nonsingular and each J_{n_i} is the n_i-by-n_i singular Jordan block. In this paper, we prove that Φ and Ψ are topologically equivalent if and only if the regularizing decompositions of their matrices coincide up to permutation of the singular summands J_{n_i} and replacement of $R \in \mathbb{C}^{r\times r}$ by a nonsingular matrix $R' \in \mathbb{C}^{r\times r}$ such that R and R' are the matrices of topologically equivalent forms $\mathbb{C}^r \times \mathbb{C}^r \to \mathbb{C}$. Analogous results for bilinear forms over $\mathbb{C}$ and over $\mathbb{R}$ are also obtained.

* Corresponding author.
E-mail addresses: carlos@sci.kuniv.edu.kw (C.M. da Fonseca), rybalkina_t@ukr.net (T. Rybalkina), sergeich@imath.kiev.ua (V.V. Sergeichuk).

http://dx.doi.org/10.1016/j.laa.2016.04.022

1. Introduction

In 1974, Gabriel [12] reduced the problem of classifying bilinear forms over an arbitrary field $\mathbb{F}$ to the problem of classifying nonsingular bilinear forms. In this paper, we take an analogous step towards the topological classification of bilinear and sesquilinear forms, reducing it to the nonsingular case.

Unlike the problem of topological classification of forms, which has not yet been considered, the problem of topological classification of linear operators has been thoroughly studied. Kuiper and Robbin [22,23] gave a criterion for topological similarity of real matrices without eigenvalues that are roots of 1. Their result was extended to complex matrices in [4]. The problem of topological similarity of matrices with an eigenvalue that is a root of 1 was also considered by these authors [22,23] as well as by Cappell and Shaneson [5–9], and by Hambleton and Pedersen [13,14]. The problem of topological classification was studied for orthogonal operators [19], for affine operators [1,3,4,10], for Möbius transformations [25], for chains of linear mappings [24], for matrix pencils [11], for oriented cycles of linear mappings [26], and for quiver representations [20].

A pair (U, Φ) consisting of a vector space U and a bilinear form Φ is called by Gabriel [12] a *bilinear space*. Similarly, we call a pair (U, Φ) a *sesquilinear space* if Φ is a sesquilinear form. A pair (U, Φ) is *singular* or *nonsingular* if Φ is so. Two spaces (U, Φ) are (V, Ψ) are *isomorphic* if there exists a linear bijection $\varphi : U \to V$ such that

$$\Phi(x, y) = \Psi(\varphi(x), \varphi(y)), \qquad \text{for all } x, y \in U. \tag{1}$$

The *direct sum* of pairs is the pair

$$(U, \Phi) \oplus (V, \Psi) := (U \oplus V, \Phi \oplus \Psi).$$

A pair is *indecomposable* if it is not isomorphic to a direct sum of pairs with vector spaces of smaller sizes.

Let vector spaces U and V be also topological spaces. For example, they are subspaces of $\mathbb{C}^m := \mathbb{C} \oplus \cdots \oplus \mathbb{C}$ (m summands) with the usual topology. We say that (U, Φ) and (V, Ψ) are *topologically equivalent* if there exists a homeomorphism $\varphi : U \to V$, i.e., a continuous bijection whose inverse is also a continuous bijection, such that (1) holds.

The main result of the paper is the following theorem, which is proved in Section 3.

Theorem 1. *Let $\mathbb{F}$ be $\mathbb{C}$ or $\mathbb{R}$. Let $(\mathbb{F}^m, \Phi)$ and $(\mathbb{F}^n, \Psi)$ be two bilinear or two sesquilinear spaces that are topologically equivalent. Suppose that*

$$(\mathbb{F}^m, \Phi) = (U_0, \Phi_0) \oplus (U_1, \Phi_1) \oplus \cdots \oplus (U_r, \Phi_r)$$

$$(\mathbb{F}^n, \Psi) = (V_0, \Psi_0) \oplus (V_1, \Psi_1) \oplus \cdots \oplus (V_s, \Psi_s),$$