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The optimal version of Hua's fundamental theorem of geometry of square matrices – the low dimensional case [☆]

Wen-ling Huang [a,b], Peter Šemrl [c,*]

[a] Department of Mathematics, University of Hamburg, Germany
[b] Department of Mathematics and Computer Sciences, University of Bremen, Germany
[c] Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, SI-1000 Ljubljana, Slovenia

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ABSTRACT

Let $\mathbb{D}$ be any division ring and p, q positive integers. The optimal version of Hua's fundamental theorem of geometry of square matrices has been known in all dimensions but the 2×2 case. We solve the remaining case by describing the general form of adjacency preserving maps $\phi : M_2(\mathbb{D}) \to M_{p\times q}(\mathbb{D})$. One of the main tools is a slight modification of known non-surjective versions of the fundamental theorem of affine geometry.

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* Corresponding author.
E-mail addresses: huang@math.uni-hamburg.de, huang@informatik.uni-bremen.de (W.-l. Huang), peter.semrl@fmf.uni-lj.si (P. Šemrl).

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1. Introduction and statement of the main result

Let $\mathbb{D}$ be a division ring and m, n positive integers. By $M_{m\times n}(\mathbb{D})$ we denote the set of all $m \times n$ matrices over $\mathbb{D}$. When $m = n$ we write $M_n(\mathbb{D}) = M_{n\times n}(\mathbb{D})$.

We consider $\mathbb{D}^n$, the set of all $1\times n$ matrices, as a left vector space over $\mathbb{D}$, and ${}^t\mathbb{D}^m$, the set of all $m \times 1$ matrices, as a right vector space over $\mathbb{D}$. The row space of $A \in M_{m\times n}(\mathbb{D})$ is defined to be the left vector subspace of $\mathbb{D}^n$ generated by the rows of A, and the row rank of A is defined to be the dimension of this subspace. Correspondingly, the column rank of A is the dimension of the column space, that is, the right vector space generated by the columns of A. These two ranks are equal for every matrix over $\mathbb{D}$ and this common value is called the rank of a matrix. It is well-known that if $\operatorname{rank} A = r$, then there exist invertible matrices $T \in M_m(\mathbb{D})$ and $S \in M_n(\mathbb{D})$ such that

$$TAS = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, \tag{1}$$

where I_r denotes the $r \times r$ identity matrix and the zeroes stand for zero matrices of the appropriate sizes. The set of matrices $M_{m\times n}(\mathbb{D})$ equipped with the so-called arithmetic distance d defined by

$$d(A, B) = \operatorname{rank}(A - B), \quad A, B \in M_{m\times n}(\mathbb{D}),$$

is a metric space. Two matrices A and B are adjacent if $d(A, B) = 1$.

Let $\mathcal{V}$ and $\mathcal{W}$ be spaces of matrices. Recall that a map $\phi : \mathcal{V} \to \mathcal{W}$ preserves adjacency in both directions if for every pair $A, B \in \mathcal{V}$ the matrices $\phi(A)$ and $\phi(B)$ are adjacent if and only if A and B are adjacent. We say that a map $\phi : \mathcal{V} \to \mathcal{W}$ preserves adjacency (in one direction only) if $\phi(A)$ and $\phi(B)$ are adjacent whenever $A, B \in \mathcal{V}$ are adjacent. The study of such maps was initiated by Hua in the series of papers [3–10], who proved the following fundamental theorem of geometry of rectangular matrices (see [18]): For every bijective map $\phi : M_{m\times n}(\mathbb{D}) \to M_{m\times n}(\mathbb{D})$, $m, n \geq 2$, preserving adjacency in both directions there exist invertible matrices $T \in M_m(\mathbb{D}), S \in M_n(\mathbb{D})$, a matrix $R \in M_{m\times n}(\mathbb{D})$, and an automorphism τ of the division ring $\mathbb{D}$ such that

$$\phi(A) = TA^\tau S + R, \quad A \in M_{m\times n}(\mathbb{D}). \tag{2}$$

Here, $A^\tau = (a_{ij})^\tau = (\tau(a_{ij}))$ is a matrix obtained from A by applying τ entrywise. In the square case $m = n$ we have the additional possibility

$$\phi(A) = T\,{}^t\big(A^\sigma\big)S + R, \quad A \in M_n(\mathbb{D}), \tag{3}$$

where T, S, R are matrices in $M_n(\mathbb{D})$ with T, S invertible, $\sigma : \mathbb{D} \to \mathbb{D}$ is an anti-automorphism, and ${}^tA$ denotes the transpose of A. Clearly, the converse statement is true as well, that is, any map of the form (2) or (3) is bijective and preserves adjacency