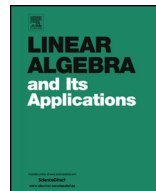




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## The optimal version of Hua's fundamental theorem of geometry of square matrices – the low dimensional case<sup>☆</sup>



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### ABSTRACT

Let  $\mathbb{D}$  be any division ring and  $p, q$  positive integers. The optimal version of Hua's fundamental theorem of geometry of square matrices has been known in all dimensions but the  $2 \times 2$  case. We solve the remaining case by describing the general form of adjacency preserving maps  $\phi : M_2(\mathbb{D}) \rightarrow M_{p \times q}(\mathbb{D})$ . One of the main tools is a slight modification of known non-surjective versions of the fundamental theorem of affine geometry.

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## 1. Introduction and statement of the main result

Let  $\mathbb{D}$  be a division ring and  $m, n$  positive integers. By  $M_{m \times n}(\mathbb{D})$  we denote the set of all  $m \times n$  matrices over  $\mathbb{D}$ . When  $m = n$  we write  $M_n(\mathbb{D}) = M_{n \times n}(\mathbb{D})$ .

We consider  $\mathbb{D}^n$ , the set of all  $1 \times n$  matrices, as a left vector space over  $\mathbb{D}$ , and  ${}^t\mathbb{D}^m$ , the set of all  $m \times 1$  matrices, as a right vector space over  $\mathbb{D}$ . The row space of  $A \in M_{m \times n}(\mathbb{D})$  is defined to be the left vector subspace of  $\mathbb{D}^n$  generated by the rows of  $A$ , and the row rank of  $A$  is defined to be the dimension of this subspace. Correspondingly, the column rank of  $A$  is the dimension of the column space, that is, the right vector space generated by the columns of  $A$ . These two ranks are equal for every matrix over  $\mathbb{D}$  and this common value is called the rank of a matrix. It is well-known that if  $\text{rank } A = r$ , then there exist invertible matrices  $T \in M_m(\mathbb{D})$  and  $S \in M_n(\mathbb{D})$  such that

$$TAS = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, \quad (1)$$

where  $I_r$  denotes the  $r \times r$  identity matrix and the zeroes stand for zero matrices of the appropriate sizes. The set of matrices  $M_{m \times n}(\mathbb{D})$  equipped with the so-called arithmetic distance  $d$  defined by

$$d(A, B) = \text{rank}(A - B), \quad A, B \in M_{m \times n}(\mathbb{D}),$$

is a metric space. Two matrices  $A$  and  $B$  are adjacent if  $d(A, B) = 1$ .

Let  $\mathcal{V}$  and  $\mathcal{W}$  be spaces of matrices. Recall that a map  $\phi : \mathcal{V} \rightarrow \mathcal{W}$  preserves adjacency in both directions if for every pair  $A, B \in \mathcal{V}$  the matrices  $\phi(A)$  and  $\phi(B)$  are adjacent if and only if  $A$  and  $B$  are adjacent. We say that a map  $\phi : \mathcal{V} \rightarrow \mathcal{W}$  preserves adjacency (in one direction only) if  $\phi(A)$  and  $\phi(B)$  are adjacent whenever  $A, B \in \mathcal{V}$  are adjacent. The study of such maps was initiated by Hua in the series of papers [3–10], who proved the following fundamental theorem of geometry of rectangular matrices (see [18]): For every bijective map  $\phi : M_{m \times n}(\mathbb{D}) \rightarrow M_{m \times n}(\mathbb{D})$ ,  $m, n \geq 2$ , preserving adjacency in both directions there exist invertible matrices  $T \in M_m(\mathbb{D})$ ,  $S \in M_n(\mathbb{D})$ , a matrix  $R \in M_{m \times n}(\mathbb{D})$ , and an automorphism  $\tau$  of the division ring  $\mathbb{D}$  such that

$$\phi(A) = TA^\tau S + R, \quad A \in M_{m \times n}(\mathbb{D}). \quad (2)$$

Here,  $A^\tau = (a_{ij})^\tau = (\tau(a_{ij}))$  is a matrix obtained from  $A$  by applying  $\tau$  entrywise. In the square case  $m = n$  we have the additional possibility

$$\phi(A) = T {}^t(A^\sigma)S + R, \quad A \in M_n(\mathbb{D}), \quad (3)$$

where  $T, S, R$  are matrices in  $M_n(\mathbb{D})$  with  $T, S$  invertible,  $\sigma : \mathbb{D} \rightarrow \mathbb{D}$  is an anti-automorphism, and  ${}^tA$  denotes the transpose of  $A$ . Clearly, the converse statement is true as well, that is, any map of the form (2) or (3) is bijective and preserves adjacency

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