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An inequality for tensor product of positive operators and its applications

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A B S T R A C T

We present an inequality for tensor product of positive operators on Hilbert spaces by considering the tensor products of operators as words on certain alphabets (i.e., a set of letters). As applications of the operator inequality and by a multilinear approach, we show some matrix inequalities concerning induced operators and generalized matrix functions (including determinants and permanents as special cases).

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1. Introduction

Let $\mathcal{H}$ be a Hilbert space over the complex number field $\mathbb{C}$ with an inner product $\langle \cdot, \cdot \rangle$. Denote by $\mathcal{B}(\mathcal{H})$ the C*-algebra of all bounded linear operators on $\mathcal{H}$. We write $A \geq 0$ if A is a positive semidefinite operator on $\mathcal{H}$ (we simply call it a positive operator), that is, A is self-adjoint and $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$. For self-adjoint $A, B \in \mathcal{B}(\mathcal{H})$, we write $A \geq B$ if $A - B \geq 0$. It is well known that if $A \geq 0$ and $B \geq 0$ then the sum $A + B \geq 0$ (on $\mathcal{H}$) and the tensor product $A \otimes B \geq 0$ (on $\otimes^2 \mathcal{H} = \mathcal{H} \otimes \mathcal{H}$). Moreover, if A is positive then the tensor product $\otimes^m A = A \otimes \cdots \otimes A$ (m copies of A) is positive (on $\otimes^m \mathcal{H}$) for any positive integer m. For the finite-dimensional case of $\mathcal{H}$, we denote by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ the smallest and largest eigenvalues of the positive linear operator (matrix) A on $\mathcal{H}$, respectively.

In Section 2, we present an inequality for positive operators with tensor product. In Section 3, as applications of our main result, we deduce inequalities for generalized matrix functions, including determinants and permanents. Our results may be regarded as additions to the recent ones in the research development of positivity (see, e.g., [1–7,14]).

2. Main results

We present our main result in this section. The proof is accomplished by using the idea of words (which, for instance, is used to show normality of matrices in [12]). Let $\{A_i, \ldots, A_j\}$ be a (multi-)set of operators from $\mathcal{B}(\mathcal{H})$ in which t operators are distinct. For example, $t = 2$ for $\{A_1, A_1, A_2\}$. A *tensor word* or *word* of operators $A_i, \ldots, A_j$ on $\mathcal{H}$ of length m and of t (distinct) representatives with respect to the tensor product $\otimes$, symbolized by $w_m^t(A_i, \ldots, A_j)$, or $w^t(A_i, \ldots, A_j)$, or even simply w^t (if no confusion is caused), is a tensor product

$$A_{s_1} \otimes \cdots \otimes A_{s_m},$$

in which $A_{s_1}, \ldots, A_{s_m}$ are taken from $\{A_i, \ldots, A_j\}$, and among $A_{s_1}, \ldots, A_{s_m}$, there are t distinct operators. For instance, $A_1 \otimes A_1 = \otimes^2 A_1$ is a w^1 word; $A_1 \otimes A_2$ is a w^2 word, and $A_3 \otimes A_2$ is also a w^2 word. Note that $w_3^2(A_1, A_2)$ may represent any of $A_1 \otimes A_1 \otimes A_2$, $A_1 \otimes A_2 \otimes A_1$, $A_2 \otimes A_1 \otimes A_1$, $A_1 \otimes A_2 \otimes A_2$, $A_2 \otimes A_1 \otimes A_2$, and $A_2 \otimes A_2 \otimes A_1$. So when we say a $w^2(A_1, A_2)$ word, we mean one of those tensor words (with the given length $m = 3$).

Theorem 1. *Let $A_1, A_2, \ldots, A_k \in \mathcal{B}(\mathcal{H})$ be positive operators. Then*

$$\otimes^m (A_1 + A_2 + \cdots + A_k) \tag{1}$$

$$- \sum_{1 \leq i_1 < \cdots < i_{k-1} \leq k} \otimes^m (A_{i_1} + A_{i_2} + \cdots + A_{i_{k-1}}) \tag{2}$$