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Geršgorin variations IV: A left eigenvector approach

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A R T I C L E   I N F O

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Dedicated to the memory of Hans Schneider, a great friend, and leader, organizer and driving force behind the International Linear Algebra Society

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A B S T R A C T

We use the left vanishing eigenvector to prove various well-known conditions for determining the nonsingularity of matrices via row sums. This is in contrast to the classical approach of using the right vanishing eigenvector. We show that on occasion this approach results in simpler proofs and generalizations of well-known results. We also present a simple proof of a generalized Gudkov's theorem.

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1. Introduction

The topic of this paper concerns necessary conditions of matrix singularity based on row sums. Typically, such results are obtained via the use of a right vanishing column eigenvector. In this paper we consider the use of the left row eigenvector to derive such conditions. For a matrix A, we use A_{ij} to denote the (i,j)-th entry of A.

Definition 1. A complex matrix A is *row diagonally dominant* if $|A_{ii}| \geq \sum_{j\neq i} |A_{ij}|$ for all i. A complex matrix A is *strictly row diagonally dominant* if $|A_{ii}| > \sum_{j\neq i} |A_{ij}|$ for all i.

2. Lévy–Desplanques theorem of matrix singularity and its generalizations

The well-known Lévy–Desplanques theorem or the equivalent Geršgorin circle criterion [4,6,10] states that strictly row diagonally dominant matrices are nonsingular:

Theorem 1. *A complex matrix A is nonsingular if $|A_{ii}| > \sum_{j\neq i} |A_{ij}|$ for all i.*

All the proofs that we are aware of proceed by assuming that A is singular with a right eigenvector z such that $Az = 0$ and work from there to reach a contradiction. The following proof uses a left eigenvector.

Proof of Theorem 1. We prove this by contradiction. Assume that A is singular and let $z \neq 0$ be a left eigenvector such that $z^T A = 0$. Then $z_j A_{jj} = -\sum_{i\neq j} z_i A_{ij}$. By the triangle inequality, we have

$$|z_j||A_{jj}| \leq \sum_{i\neq j} |z_i||A_{ij}| \tag{2.1}$$

Summing over j, we get

$$\sum_j |z_j||A_{jj}| \leq \sum_{i,j}^{i\neq j} |z_j||A_{ji}| = \sum_j |z_j| \sum_{i\neq j} |A_{ji}| \tag{2.2}$$

Since $|A_{jj}| > \sum_{i\neq j} |A_{ji}|$, this implies that $|z_j| = 0$ for all j which contradicts the fact that $z \neq 0$. □

It should be pointed out that the Camion–Hoffman theorem, which is the converse of a (generalized) Lévy–Desplanques theorem, is proved in [3] using the left vanishing row eigenvector as well. Taussky [15] proved the following generalization of Theorem 1:

Theorem 2. *An irreducible complex matrix A is nonsingular if $|A_{ii}| \geq \sum_{j\neq i} |A_{ij}|$ for all i with the inequality strict for at least one i.*