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Preservers of unitary similarity functions on Lie products of matrices

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ABSTRACT

Denote by M_n the set of $n \times n$ complex matrices. Let $f : M_n \to [0, \infty)$ be a continuous map such that $f(\mu UAU^*) = f(A)$ for any complex unit μ, $A \in M_n$ and unitary $U \in M_n$, $f(X) = 0$ if and only if $X = 0$ and the induced map $t \mapsto f(tX)$ is monotonically increasing on $[0, \infty)$ for any rank one nilpotent $X \in M_n$. Characterization is given for surjective maps ϕ on M_n satisfying $f(AB - BA) = f(\phi(A)\phi(B) - \phi(B)\phi(A))$. The general theorem is then used to deduce results on special cases when the function is the pseudo spectrum and the pseudo spectral radius.

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1. Introduction

Let M_n be the set of $n \times n$ matrices. A function $f : M_n \to \mathbb{R}$ is a *radial unitary similarity invariant function* if

(P1) $f(\mu UAU^*) = f(A)$ for a complex unit μ, $A \in M_n$ and unitary $U \in M_n$.

In [11], the authors studied unitary similarity invariant functions that are norms on M_n, and determined the structure of maps $\phi : M_n \to M_n$ satisfying

$$f(AB - BA) = f(\phi(A)\phi(B) - \phi(B)\phi(A)) \quad \text{for all } A, B \in M_n. \tag{1.1}$$

In [11, Remark 2.7], it was pointed out that the result actually holds for more general unitary similarity invariant functions. However, no detail was given, and it is not straightforward to apply the results to a specific problem. For instance, it is unclear how one can apply the result to study preservers of pseudo spectrum of Lie product of matrices[1]; see the definition in Section 3. To fill this gap, we extend the result in [11] to continuous radial unitary similarity invariant functions $f : M_n \to \mathbb{R}$ satisfying the following properties.

(P2) For any $X \in M_n$ we have $f(X) = f(0_n)$ if and only if $X = 0_n$, the $n \times n$ zero matrix.

(P3) For any rank one nilpotent $X \in M_n$, the map $t \mapsto f(tX)$ on $[0, \infty)$ is strictly increasing.

For a function $f : M_n \to [0, \infty)$ satisfying (P1)–(P3), we show that if $\phi : M_n \to M_n$ is a surjective map satisfying (1.1), then there is a unitary $U \in M_n$ and a subset $\mathcal{N}_n$ of normal matrices in M_n such that ϕ has the form

$$\phi(A) = \begin{cases} \mu_A U A^\dagger U^* + \nu_A I_n & A \in M_n \setminus \mathcal{N}_n \\ \mu_A U (A^\dagger)^* U^* + \nu_A I_n & A \in \mathcal{N}_n, \end{cases}$$

where

(a) $\mu_A, \nu_A \in \mathbb{C}$ with $|\mu_A| = 1$, depending on A,
(b) $A^\dagger = A$, $\overline{A}$, A^t or A^*, and
(c) $\mathcal{N}_n$ depends on the given unitarily invariant function f.

[1] This is a question raised by Professor Molnar to the second and third authors at the 2014 Summer Conference of the Canadian Mathematics Society.