Linear Algebra and its Applications 498 (2016) 181–200

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

The enhanced principal rank characteristic sequence

Steve Butler [a], Minerva Catral [b], Shaun M. Fallat [c,1], H. Tracy Hall [d], Leslie Hogben [a,e,*], P. van den Driessche [f,2], Michael Young [a]

[a] *Department of Mathematics, Iowa State University, Ames, IA 50011, USA*
[b] *Department of Mathematics and Computer Science, Xavier University, Cincinnati, OH 45207, USA*
[c] *Department of Mathematics and Statistics, University of Regina, Regina, SK, S4S 0A2, Canada*
[d] *Department of Mathematics, Brigham Young University, Provo, UT 84602, USA*
[e] *American Institute of Mathematics, 600 E. Brokaw Road, San Jose, CA 95112, USA*
[f] *Department of Mathematics and Statistics, University of Victoria, Victoria, BC, V8W 2Y2, Canada*

A R T I C L E I N F O

Article history:
Received 31 December 2013
Accepted 22 March 2015
Available online 3 April 2015
Submitted by R. Brualdi

MSC:
15A15
15A03
15B57
05C50

A B S T R A C T

The enhanced principal rank characteristic sequence (epr-sequence) of a symmetric $n \times n$ matrix is a sequence $\ell_1\ell_2\cdots\ell_n$ where ℓ_k is A, S, or N according as all, some, or none of its principal minors of order k are nonzero. Such sequences give more information than the (0,1) pr-sequences previously studied (where basically the kth entry is 0 or 1 according as none or at least one of its principal minors of order k is nonzero). Various techniques including the Schur complement are introduced to establish that certain subsequences such as NAN are forbidden in epr-sequences over

* Corresponding author.
E-mail addresses: butler@iastate.edu (S. Butler), catralm@xavier.edu (M. Catral), shaun.fallat@uregina.ca (S.M. Fallat), H.Tracy@gmail.com (H.T. Hall), LHogben@iastate.edu, hogben@aimath.org (L. Hogben), pvdd@math.uvic.ca (P. van den Driessche), myoung@iastate.edu (M. Young).

[1] Research supported in part by NSERC Discovery grant RGPIN 2014-06036.
[2] Research supported in part by NSERC Discovery grant 8965.

http://dx.doi.org/10.1016/j.laa.2015.03.023

Keywords:
Principal rank characteristic sequence
Enhanced principal rank characteristic sequence
Minor
Rank
Symmetric matrix
Hermitian matrix
Schur complement

fields of characteristic not two. Using probabilistic methods over fields of characteristic zero, it is shown that any sequence of As and Ss ending in A is attainable, and any sequence of As and Ss followed by one or more Ns is attainable; additional families of attainable epr-sequences are constructed explicitly by other methods. For real symmetric matrices of orders 2, 3, 4, and 5, all attainable epr-sequences are listed with justifications.

1. Introduction

For a symmetric matrix over a field F or a complex Hermitian matrix, Brualdi et al. [2] and Barrett et al. [1] considered a principal rank characteristic sequence, which records with a 1 or a 0 whether or not there is a full rank principal submatrix of each order. More precisely, the *principal rank characteristic sequence* of an $n \times n$ symmetric or complex Hermitian matrix B is the sequence $\mathrm{pr}(B) = r_0]r_1r_2 \cdots r_n$, where for $k = 0, 1, \ldots, n$, a 1 in the kth position indicates the existence of a principal submatrix of rank k and a 0 indicates no such submatrix exists. To obtain more information, we refine this sequence and instead of considering the presence or absence of such a principal submatrix, we consider three possibilities in the following definition.

Definition 1.1. The *enhanced principal rank characteristic sequence* of a symmetric matrix $B \in F^{n\times n}$ (or Hermitian matrix $B \in \mathbb{C}^{n\times n}$) is the sequence (epr-sequence) $\mathrm{epr}(B) = \ell_1\ell_2 \cdots \ell_n$ where

$$\ell_k = \begin{cases} \mathtt{A} & \text{if all } k \times k \text{ principal minors of the given order are nonzero;} \\ \mathtt{S} & \text{if some but not all } k \times k \text{ principal minors are nonzero;} \\ \mathtt{N} & \text{if none of the } k \times k \text{ principal minors are nonzero, i.e., all are zero.} \end{cases}$$

We are interested in which epr-sequences are *attainable* over a given field F, i.e., can be attained by some (symmetric or Hermitian) matrix over F, and also which sequences are *forbidden* over a given field, i.e., no such matrix has the sequence. We can now drop the convention of having a 0th term given by r_0 in the pr-sequence. In particular the relationship between the old and new naming conventions for the beginning of a sequence is as follows: $1]0 \leftrightarrow \mathtt{N}$, $1]1 \leftrightarrow \mathtt{S}$, and $0]1 \leftrightarrow \mathtt{A}$.

Brualdi et al. [2] introduced the definition of a pr-sequence for a real symmetric matrix as a simplification of the principal minor assignment problem as stated in [4]. The study of epr-sequences provides additional information that may be helpful in work on the principal minor assignment problem, while remaining somewhat tractable. Furthermore, the enhanced principal rank characteristic sequence can be used to answer the following question [5, p. 112]: "For a real symmetric matrix, which lists of sizes, for which there exists a singular principal submatrix, can occur?" (See Corollary 4.7.)

In Section 2, we identify certain forbidden and certain attainable epr-sequences, with some results depending on the field; the Schur complement method for establishing