Linear Algebra and its Applications 498 (2016) 219–230

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Completely mixed linear games on a self-dual cone

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A R T I C L E I N F O

Article history:
Received 12 December 2014
Accepted 18 May 2015
Available online 29 May 2015
Submitted by R. Brualdi

This paper is dedicated to Hans Schneider with profound respect, admiration, and appreciation

MSC:
91A05
46N10
17C20

Keywords:
Self-dual cone
Strategy
Value
Completely mixed game
Symmetric cone
Z-transformation

A B S T R A C T

Given a finite dimensional real inner product space V with a self-dual cone K, an element e in K° (the interior of K), and a linear transformation L on V, the value of the linear game (L, e) is defined by

$$v(L, e) := \max_{x \in \Delta(e)} \min_{y \in \Delta(e)} \langle L(x), y \rangle = \min_{y \in \Delta(e)} \max_{x \in \Delta(e)} \langle L(x), y \rangle,$$

where $\Delta(e) = \{x \in K : \langle x, e \rangle = 1\}$. In [5], various properties of a linear game and its value were studied and some classical results of Kaplansky [6] and Raghavan [8] were extended to this general setting. In the present paper, we study how the value and properties change as e varies in K°. In particular, we study the structure of the set $\Omega(L)$ of all e in K° for which the game (L, e) is completely mixed and identify certain classes of transformations for which $\Omega(L)$ equals K°. We also describe necessary and sufficient conditions for a game (L, e) to be completely mixed when $v(L, e) = 0$, thereby generalizing a result of Kaplansky [6].

1. Introduction

This paper is a continuation of [5], where the concept of value of a (zero-sum) matrix game is generalized to a linear transformation defined on a self-dual cone in a real finite

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http://dx.doi.org/10.1016/j.laa.2015.05.025

dimensional Hilbert space. To elaborate, consider a finite dimensional real inner product space $(V, \langle\cdot\,,\cdot\rangle)$ and a self-dual cone K in V. We fix an element e in K° (the interior of K) and let

$$\Delta(e) := \{x \in K : \langle x, e\rangle = 1\}, \tag{1}$$

the elements of which are called 'strategies'. Given a linear transformation L from V to V, the zero-sum linear game — denoted by (L, e) — is played by two players I and II in the following way: If player I chooses strategy $x \in \Delta(e)$ and player II chooses strategy $y \in \Delta(e)$, then the pay-off for player I is $\langle L(x), y\rangle$ and the pay-off for player II is $-\langle L(x), y\rangle$. Both players try to maximize their pay-offs. Since $\Delta(e)$ is a compact convex set and L is linear, by the min–max Theorem of von Neumann (see [7, Theorems 1.5.1 and 1.3.1]), there exist *optimal strategies* $\bar{x}$ for player I and $\bar{y}$ for player II which satisfy

$$\langle L(x), \bar{y}\rangle \le \langle L(\bar{x}), \bar{y}\rangle \le \langle L(\bar{x}), y\rangle \quad \forall\, x, y \in \Delta(e). \tag{2}$$

This means that players I and II do not gain by unilaterally changing their strategies from the optimal strategies $\bar{x}$ and $\bar{y}$. The number

$$v(L, e) := \langle L(\bar{x}), \bar{y}\rangle$$

is the *value of the game*, or simply, *the value of* (L, e). The pair $(\bar{x}, \bar{y})$ is called an *optimal strategy pair* for (L, e). We note that $v(L, e)$ is also given by [7, Theorems 1.5.1 and 1.3.1]

$$v(L, e) = \max_{x\in\Delta(e)} \min_{y\in\Delta(e)} \langle L(x), y\rangle = \min_{y\in\Delta(e)} \max_{x\in\Delta(e)} \langle L(x), y\rangle. \tag{3}$$

We say that the game (L, e) is *completely mixed* if for every optimal strategy pair $(\bar{x}, \bar{y})$ of (L, e), $\bar{x}$ and $\bar{y}$ belong to K°. The above concepts and definitions reduce to the classical ones when $V = R^n$ (with the usual inner product), $K = R^n_+$ (the nonnegative orthant), $L \in R^{n\times n}$ and e is the (column) vector of ones. In [5], several classical results of Kaplansky [6] and Raghavan [8] were extended to this general setting and their connections to dynamical systems were explored. As in the classical case, the uniqueness of the optimal strategy pair prevails when the game is completely mixed (see Theorem 4 in [5]). The completely mixed property was investigated for **Z**, Lyapunov-like and Stein-like transformations (see Section 2 for definitions). In particular, it was shown in [5] that the game (L, e) is completely mixed when L is a **Z**-transformation with $v(L, e) > 0$ or L is a Lyapunov/Stein-like transformation with $v(L, e) \neq 0$.

In the present paper, we address three issues: (i) How the value and the optimal strategies change as L and e are changed, (ii) how the value changes under cone automorphisms, and (iii) how, for a given transformation L, the completely mixed property changes as e varies over the interior of K.

Addressing (i), we show that the value varies continuously and the optimal strategy set is upper semicontinuous in L and e. We also specify (upper) bounds for $v(L, e)$.