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Strong shift equivalence and the Generalized Spectral Conjecture for nonnegative matrices

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ABSTRACT

Given matrices A and B shift equivalent over a dense subring $\mathcal{R}$ of $\mathbb{R}$, with A primitive, we show that B is strong shift equivalent over $\mathcal{R}$ to a primitive matrix. This result shows that the weak form of the Generalized Spectral Conjecture for primitive matrices implies the strong form. The foundation of this work is the recent result that for any ring $\mathcal{R}$, the group $\mathrm{NK}_1(\mathcal{R})$ of algebraic K-theory classifies the refinement of shift equivalence by strong shift equivalence for matrices over $\mathcal{R}$.

1. Introduction

The purpose of this paper[1] is to prove the following theorem and explain its context.

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[1] This paper is an outgrowth of the paper [2], for which its authors were awarded (along with Robert Thompson) the second Hans Schneider Prize.

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Theorem 1.1. *Suppose $\mathcal{R}$ is a dense subring of $\mathbb{R}$, A is a primitive matrix over $\mathcal{R}$ and B is a matrix over $\mathcal{R}$ which is shift equivalent over $\mathcal{R}$ to A.*

Then B is strong shift equivalent over $\mathcal{R}$ to a primitive matrix.

We begin with the context. By ring, we mean a ring with 1; by a semiring, we mean a semiring containing $\{0, 1\}$. A primitive matrix is a square matrix which is nonnegative (meaning entrywise nonnegative) such that for some $k > 0$ its kth power is a positive matrix. Definitions and more background for shift equivalence (SE) and strong shift equivalence (SSE) are given in Section 2.

We recall the Spectral Conjecture for primitive matrices from [2]. In the statement, $\Delta = (d_1, \dots, d_k)$ is a k-tuple of nonzero complex numbers. Δ is the *nonzero spectrum* of a matrix A if A has characteristic polynomial of the form $\chi_A(t) = t^m \prod_{1\le i\le k}(t - d_i)$. Δ has a *Perron value* if there exists i such that $d_i > |d_j|$ when $j \neq i$. The *trace* of Δ is $\mathrm{tr}(\Delta) = d_1 + \cdots + d_k$. Δ^n denotes $((d_1)^n, \dots, (d_k)^n)$, the tuple of nth powers; and the *nth net trace* of Δ is

$$\mathrm{tr}_n(\Delta) = \sum_{d|n} \mu(n/d)\mathrm{tr}(\Delta^d)$$

in which μ is the Möbius function ($\mu(1) = 1$; $\mu(n) = (-1)^r$ if n is the product of r distinct primes; $\mu(n) = 0$ if n is divisible by the square of a prime).

Spectral Conjecture 1.2. *(See [2].) Let $\mathcal{R}$ be a subring of $\mathbb{R}$. Then Δ is the nonzero spectrum of some primitive matrix over $\mathcal{R}$ if and only if the following conditions hold:*

(1) Δ has a Perron value.
(2) The coefficients of the polynomial $\prod_{i=1}^{k}(t - d_i)$ lie in $\mathcal{R}$.
(3) If $\mathcal{R} = \mathbb{Z}$, then for all positive integers n, $\mathrm{tr}_n(\Delta) \ge 0$;
if $\mathcal{R} \neq \mathbb{Z}$, then for all positive integers n and k,
(i) $\mathrm{tr}(\Delta^n) \ge 0$ and (ii) $\mathrm{tr}(\Delta^n) > 0$ implies $\mathrm{tr}(\Delta^{nk}) > 0$.

It is not difficult to check that the nonzero spectrum of a primitive matrix satisfies the three conditions [2]. (We remark, following [8] it is known that the nonzero spectra of symmetric primitive matrices cannot possibly have such a simple characterization.)

To understand the possible spectra of nonnegative matrices is a classical problem of linear algebra (for early background see e.g. [2]) on which interesting progress continues (see e.g. [7,14,15,13] and their references). Understanding the nonzero spectra of primitive matrices is a variant of this problem and also an approach to it: to know the minimal size of a primitive matrix with a prescribed nonzero spectrum is to solve the classical problem (for details, see [2]); and it is in the primitive case that the Perron–Frobenius constraints manifest most simply.