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Geometric mapping properties of semipositive matrices

M.J. Tsatsomeros

Mathematics Department, Washington State University, Pullman, WA 99164, United States

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ABSTRACT

Semipositive matrices map a positive vector to a positive vector and as such they are a very broad generalization of the irreducible nonnegative matrices. Nevertheless, the ensuing geometric mapping properties of semipositive matrices result in several parallels to the theory of cone preserving and cone mapping matrices. It is shown that for a semipositive matrix A, there exist a proper polyhedral cone K_1 of nonnegative vectors and a polyhedral cone K_2 of nonnegative vectors such that $AK_1 = K_2$. The set of all nonnegative vectors mapped by A to the nonnegative orthant is a proper polyhedral cone; as a consequence, A belongs to a proper polyhedral cone comprising semipositive matrices. When the powers A^k $(k = 0, 1, \ldots)$ have a common semipositivity vector, then A has a positive eigenvalue. If A has a sole peripheral eigenvalue λ and the powers of A have a common semipositivity vector with a non-vanishing term in the direction of the left eigenspace of λ, then A leaves a proper cone invariant.

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E-mail address: tsat@wsu.edu.

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1. Introduction

By its definition, a semipositive matrix maps a positive vector to another positive vector. This is a trait shared by many important matrix classes, such as the nonnegative matrices, the positive definite matrices, the M-matrices and the P-matrices. The first systematic consideration of semipositive matrices occurs in [3], under the name 'class S'. The basic properties of semipositive matrices, their subclasses and applications can be reviewed in [2,3,5].

It is enticing to study the characteristic property of semipositive matrices as a mapping property between geometric objects, specifically, convex cones in $\mathbb{R}^n$. Matrix semipositivity can indeed be viewed as the broadest possible generalization of the irreducible nonnegative matrices which map all positive vectors to positive vectors. As such, one would not expect that many strong properties of nonnegative matrices generalize to semipositive matrices. There are, however, some cone-theoretic and Perron–Frobenius-type implications of semipositivity, especially when all the powers of a matrix are assumed or implied to be semipositive.

Our effort to reveal geometric and spectral consequences of semipositivity unfolds as follows: Section 2 contains general notation and definitions. Section 3 examines three factorizations of semipositive matrices into semipositive matrices with common semipositivity vectors. Section 4 contains some geometric observations regarding semipositive maps and examines the set of all nonnegative vectors that get mapped by a semipositive matrix to the nonnegative orthant. In Section 5, spectral results related to semipositivity are collected, in particular, as to when a semipositive matrix has a positive eigenvalue with a corresponding nonnegative eigenvector.

2. Notation and definitions

The all-ones vector is denoted by $e = [1\ 1\ \dots\ 1]^T$ with its size determined by the context.

The $n \times n$ matrices with complex (resp., real) entries is denoted by $M_n(\mathbb{C})$ (resp., $M_n(\mathbb{R})$). The following notation is used for $A \in M_n(\mathbb{C})$:

The *spectrum* of A is $\sigma(A)$, viewed as a multiset containing the eigenvalues according to their multiplicities. The *spectral radius* of A is $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$. Eigenvalues λ with $|\lambda| = \rho(A)$ are referred to as *peripheral*. The *index* of $\lambda \in \sigma(A)$ is the size of the largest Jordan block of λ in the Jordan canonical form of A and is denoted by ν_λ. By convention, $\nu_\lambda = 0$ if and only if $\lambda \notin \sigma(A)$.

For $\alpha \subseteq \{1, 2, \dots, n\}$, $\bar{\alpha} = \{1, 2, \dots, n\} \setminus \alpha$. $A[\alpha, \beta]$ is the submatrix of A whose rows and columns are indexed by $\alpha, \beta \subseteq \{1, 2, \dots, n\}$, respectively; the elements of α, β are assumed to be in ascending order. When a row or column index set is empty, the corresponding submatrix is considered vacuous and by convention has determinant equal to 1. $A[\alpha, \alpha]$ is abbreviated by $A[\alpha]$.