Linear Algebra and its Applications 498 (2016) 366–377

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

The enhanced principal rank characteristic sequence for skew-symmetric matrices

Shaun M. Fallat [a,*,1], Dale D. Olesky [b,1], Pauline van den Driessche [c,1]

[a] *Department of Mathematics and Statistics, University of Regina, Regina, SK, Canada*
[b] *Department of Computer Science, University of Victoria, Victoria, BC, V8W 2Y2, Canada*
[c] *Department of Mathematics and Statistics, University of Victoria, Victoria, BC, V8W 2Y2, Canada*

A R T I C L E I N F O

Article history:
Received 24 June 2015
Accepted 15 August 2015
Available online 29 August 2015
Submitted by R. Brualdi

MSC:
15A15
15A03
15B57

Keywords:
Principal rank characteristic sequence
Enhanced principal rank characteristic sequence
Minor
Rank
Skew-symmetric matrix

A B S T R A C T

The enhanced principal rank characteristic sequence (epr-sequence) was originally defined for an $n \times n$ real symmetric matrix or an $n \times n$ Hermitian matrix. Such a sequence is defined to be $\ell_1\ell_2\cdots\ell_n$ where ℓ_k is A, S, or N depending on whether all, some, or none of the matrix principal minors of order k are nonzero. Here we give a complete characterization of the attainable epr-sequences for real skew-symmetric matrices. With the constraint that $\ell_k = 0$ if k is odd, we show that nearly all epr-sequences are attainable by skew-symmetric matrices, which is in contrast to the case of real symmetric or Hermitian matrices for which many epr-sequences are forbidden.

* Corresponding author.
E-mail addresses: shaun.fallat@uregina.ca (S.M. Fallat), dolesky@cs.uvic.ca (D.D. Olesky), pvdd@math.uvic.ca (P. van den Driessche).

[1] Research supported in part by an NSERC Discovery grants—Fallat: RGPIN-2014-06036, Olesky: Dis. Grant 8214, van den Driessche: Dis. Grant 8965.

http://dx.doi.org/10.1016/j.laa.2015.08.014
0024-3795/© 2015 Elsevier Inc. All rights reserved.

1. Introduction

For a symmetric matrix over a field F or a complex Hermitian matrix, Brualdi et al. [2] and Barrett et al. [1] defined and studied the principal rank characteristic sequence, which records with a 1 or a 0 whether or not there is a full rank principal submatrix of each order. This concept was extended (or enhanced) in [3] by broadening the measure contained within the elements of a principal rank characteristic sequence. Accordingly this sequence considers the presence or absence of such a principal submatrix, via three possibilities in the following definition.

Definition 1.1. The *enhanced principal rank characteristic sequence* of a matrix $B \in F^{n\times n}$ is the sequence (epr-sequence) $\operatorname{epr}(B) = \ell_1\ell_2\cdots\ell_n$ where

$$\ell_k = \begin{cases} \mathtt{A} & \text{if all } k\times k \text{ principal minors of the given order are nonzero;} \\ \mathtt{S} & \text{if some but not all } k\times k \text{ principal minors are nonzero;} \\ \mathtt{N} & \text{if none of the } k\times k \text{ principal minors are nonzero, i.e., all are zero.} \end{cases}$$

We are interested in which epr-sequences are *attainable* by skew-symmetric matrices over $\mathbb{R}$, that is, are realized as the epr-sequence of some skew-symmetric matrix.

Brualdi et al. [2] introduced the definition of a pr-sequence for a real symmetric matrix as a simplification of the principal minor assignment problem as stated in [6]. The study of epr-sequences provides additional information that may be helpful in work on the principal minor assignment problem, while remaining somewhat tractable.

For $B \in \mathbb{R}^{n\times n}$ and $\alpha, \beta \subseteq \{1, 2, \ldots, n\}$, the submatrix of B lying in rows indexed by α and columns indexed by β is denoted by $B[\alpha, \beta]$. Further, the complementary submatrix obtained from B by deleting the rows indexed by α and columns indexed by β is denoted by $B(\alpha, \beta)$. If $\alpha = \beta$, then the principal submatrix $B[\alpha, \alpha]$ is abbreviated to $B[\alpha]$, while the complementary principal submatrix is denoted by $B(\alpha)$. The complement of α is denoted by α^c.

For $B \in \mathbb{R}^{n\times n}$, we let $\sigma(B)$ denote the multiset of eigenvalues of B, and we denote the rank of B by $\operatorname{rank}(B)$. Following the notation in [1], we let $\overline{\ell_i \cdots \ell_j}$ indicate that the (complete) sequence may be repeated as many times as necessary (or may be omitted entirely). All matrices considered here are skew-symmetric over $\mathbb{R}$ and of order ≥ 2 (since for $n = 1$ the only case is $\operatorname{epr}(0) = \mathtt{N}$).

1.1. Basic facts on the epr-sequences of skew-symmetric matrices

This section contains some key fundamental facts on skew-symmetric matrices that we connect to properties of their epr-sequences. The fact that a real matrix satisfies $B = -B^T$ places significant restrictions on the spectrum of B, the rank of B, and the quadratic form x^TBx, for $x \in \mathbb{R}^n$. For instance, as is well known, if $Bx = \lambda x$ and $x \neq 0$, then $\lambda = ik$ for some $k \in \mathbb{R}$, i.e., λ is pure imaginary. Hence, if $\lambda \in \sigma(B) \cap \mathbb{R}$, then