Linear Algebra and its Applications 498 (2016) 553–573

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

On the extreme points of quantum channels

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ARTICLE INFO

Article history:
Received 2 September 2015
Accepted 1 February 2016
Available online 10 February 2016
Submitted by R. Brualdi

MSC:
15B48
47B65
94A17
94A40

Keywords:
Extreme point
Quantum channel
Additivity conjectures

ABSTRACT

Let $\mathcal{L}_{m,n}$ denote the convex set of completely positive trace preserving operators from $\mathbb{C}^{m\times m}$ to $\mathbb{C}^{n\times n}$, i.e. quantum channels. We give a necessary condition for $L \in \mathcal{L}_{m,n}$ to be an extreme point. We show that generically, this condition is also sufficient. We characterize completely the extreme points of $\mathcal{L}_{2,2}$ and $\mathcal{L}_{3,2}$, i.e. quantum channels from qubits to qubits and from qutrits to qubits.

1. Introduction

Denote by $\mathbb{C}^{m\times n}$, $\mathrm{H}_m \supset \mathrm{H}_{m,+} \supset \mathrm{H}_{m,+,1}$ the space of complex $m \times n$ matrices, $m \times m$ hermitian matrices, the cone of nonnegative definite matrices and the set of density matrices respectively. For $A \in \mathbb{C}^{m\times m}$ we denote $A \geq 0$ if and only if $A \in \mathrm{H}_{m,+}$. Denote

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[1] Supported by National Science Foundation grant DMS-1216393.

http://dx.doi.org/10.1016/j.laa.2016.02.001
0024-3795/

by $\mathcal{P}_m \subset \mathrm{H}_{m,+,1}$ the set of all pure states in $\mathrm{H}_{m,+,1}$, i.e. all rank one hermitian matrices of order m with trace one. Let $[m] = \{1, \ldots, m\}$ for any positive integer m. Recall that $L : \mathbb{C}^{m\times m} \to \mathbb{C}^{n\times n}$ is called completely positive if

$$L(X) = \sum_{i=1}^{k} A_i X A_i^*, \quad A_i \in \mathbb{C}^{n\times m}, i \in [k]. \tag{1.1}$$

Observe that

$$L(X) = \sum_{i=1}^{k} B_i X B_i^*, \quad B_i = \zeta_i A_i,\ |\zeta_i| = 1,\ i \in [k]. \tag{1.2}$$

L is called quantum channel if L is completely positive and $L : \mathrm{H}_{m,+,1} \to \mathrm{H}_{n,+,1}$. This is equivalent to the statement that

$$\sum_{i=1}^{k} A_i^* A_i = I_m, \tag{1.3}$$

i.e. L is trace preserving: $\operatorname{tr} L(X) = \operatorname{tr} X$ for all $X \in \mathbb{C}^{m\times m}$. Denote by $\mathcal{L}_{m,n}$ the convex set of all quantum channels $L : \mathrm{H}_{m,+,1} \to \mathrm{H}_{n,+,1}$. The aim of this note is to study the extreme points of $\mathcal{L}_{m,n}$. We reprove and extend some of the results in [2,11,10]. Some related results are discussed in [1,13,14,4]. One of the novel features of this paper is the use of the notions and results of complex and real algebraic geometry, and semi-algebraic geometry.

The paper is organized as follows. In Section 2 we state Choi's theorem, characterizing a completely positive operator $L : \mathbb{C}^{m\times m} \to \mathbb{C}^{n\times n}$ in terms of a suitable matrix representation $Z(L)$ in $\mathbb{C}^{mn\times mn}$, and point out a relation between rank $Z(L)$ and the number of summands in the representation (1.1) of L. In Section 3 extreme points of the compact convex set $\mathcal{L}_{m,n}$ are considered. We give a necessary and sufficient condition for L to be an extreme point of $\mathcal{L}_{m,n}$ in terms of the null space of the matrix $Z(L)$. For this purpose we give a new proof of the result stating that if $L \in \mathcal{L}_{m,n}$ is an extreme point then rank $Z(L) \in [m]$.

Conversely, in Section 4 we show that a generic $L \in \mathcal{L}_{m,n}$ with rank $Z(L) = m$ is an extreme point of $\mathcal{L}_{m,n}$. Information on L when rank $Z(L) \le m$ is also given. In Section 5 we give a dimension condition on $L \in \mathcal{L}_{m,n}$ so that $L(\mathrm{H}_{m,+,1})$ contains a density matrix of rank at most p. In Section 6 we fully characterize the extreme points of $\mathcal{L}_{2,2}$, namely we show that $L \in \mathcal{L}_{2,2}$ is an extreme point of that set if and only if either L is a unitary similarity transformation, or rank $Z(L) = 2$ and L is not a convex combination of two distinct unitary similarity transformations. This is known, but our approach is new. In Section 7 we characterize the extreme points of $\mathcal{L}_{3,2}$. Section 8 contains a brief discussion of entropy of quantum channels.