Linear Algebra and its Applications 498 (2016) 592–627

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Non-archimedean valuations of eigenvalues of matrix polynomials

Marianne Akian [a,*,1], Ravindra Bapat [b,2], Stéphane Gaubert [a,1]

[a] *INRIA and CMAP, École polytechnique, UMR 7641, CNRS, Route de Saclay, 91128 Palaiseau Cedex, France*
[b] *Indian Statistical Institute, New Delhi, 110016, India*

A R T I C L E   I N F O

Article history:
Received 1 March 2015
Accepted 29 February 2016
Available online 19 March 2016
Submitted by R. Brualdi

We dedicate this article to the memory of Hans Schneider

MSC:
47A55
47A75
05C50
12K10

Keywords:
Perturbation theory
Max-plus algebra
Tropical semifield
Spectral theory
Newton–Puiseux theorem
Amoeba
Majorization

A B S T R A C T

We establish general weak majorization inequalities, relating the leading exponents of the eigenvalues of matrices or matrix polynomials over the field of Puiseux series with the tropical analogues of eigenvalues. We also show that these inequalities become equalities under genericity conditions, and that the leading coefficients of the eigenvalues are determined as the eigenvalues of auxiliary matrix polynomials.

* Corresponding author.
E-mail addresses: Marianne.Akian@inria.fr (M. Akian), rbb@isid1.isid.ac.in (R. Bapat), Stephane.Gaubert@inria.fr (S. Gaubert).

[1] The first and third authors were partially supported by PGMO, a joint program of EDF and FMJH (Fondation Mathématique Jacques Hadamard).
[2] The second author acknowledges support from the JC Bose Fellowship, Department of Science and Technology, Government of India.

http://dx.doi.org/10.1016/j.laa.2016.02.036

Graphs
Optimal assignment

1. Introduction

1.1. Non-archimedean valuations and tropical geometry

A non-archimedean valuation ν on a field $\mathbb{K}$ is a map $\mathbb{K} \to \mathbb{R} \cup \{+\infty\}$ such that

$$\nu(a) = +\infty \iff a = 0 \tag{1a}$$

$$\nu(a+b) \geq \min(\nu(a), \nu(b)) \tag{1b}$$

$$\nu(ab) = \nu(a) + \nu(b) \ . \tag{1c}$$

These properties imply that $\nu(a+b) = \min(\nu(a), \nu(b))$ for $a, b \in \mathbb{K}$ such that $\nu(a) \neq \nu(b)$. Therefore, the map ν is almost a morphism from $\mathbb{K}$ to the *min-plus* or *tropical* semifield $\mathbb{R}_{\min}$, which is the set $\mathbb{R} \cup \{+\infty\}$, equipped with the addition $(a, b) \mapsto \min(a, b)$ and the multiplication $(a, b) \mapsto a+b$. A basic example of field with a non-archimedean valuation is the field of complex Puiseux series, with the valuation which takes the leading (smallest) exponent of a series. The images by a non-archimedean valuation of algebraic subsets of $\mathbb{K}^n$ are known as *non-archimedean amoebas*. The latter have a combinatorial structure which is studied in tropical geometry [30,42]. For instance, Kapranov's theorem shows that the closure of the image by a non-archimedean valuation of an algebraic hypersurface over an algebraically closed field is a tropical hypersurface, i.e., the non-differentiability locus of a convex polyhedral function, see [23]. This generalizes the characterization of the leading exponents of the different branches of an algebraic curve in terms of the slopes of the Newton polygon, which is part of the classical Newton–Puiseux theorem.

1.2. Main results

In the present paper, we consider the eigenproblem over the field of complex Puiseux series and related fields of functions. Our aim is to relate the images of the eigenvalues by the non-archimedean valuation with certain easily computable combinatorial objects called *tropical eigenvalues*.

The first main result of the present paper, Theorem 4.4, shows that the sequence of valuations of the eigenvalues of a matrix $\mathcal{A} \in \mathbb{K}^{n\times n}$ is weakly (super) majorized by the sequence of (algebraic) tropical eigenvalues of the matrix obtained by applying the valuation to the entries of $\mathcal{A}$. Next, we show that the same majorization inequality holds under more general circumstances. In particular, in Theorem 5.2 and Corollary 5.3, we deal with a relaxed definition of the valuation. In the spirit of large deviations theory, we assume that the entries of the matrix are functions of a small parameter ϵ, which may not have a Puiseux series type expansion, but have some mild form of first order