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Null ideals of matrices over residue class rings of principal ideal domains

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ABSTRACT

Given a square matrix A with entries in a commutative ring S, the ideal of $S[X]$ consisting of polynomials f with $f(A) = 0$ is called the null ideal of A. Very little is known about null ideals of matrices over general commutative rings. First, we determine a certain generating set of the null ideal of a matrix in case $S = {}^{D}/_{dD}$ is the residue class ring of a principal ideal domain D modulo $d \in D$. After that we discuss two applications. We compute a decomposition of the S-module $S[A]$ into cyclic S-modules and explain the strong relationship between this decomposition and the determined generating set of the null ideal of A. And finally, we give a rather explicit description of the ring $\mathrm{Int}(A, \mathrm{M}_n(D))$ of all integer-valued polynomials on A.

1. Introduction

Matrices with entries in commutative rings arise in numerous contexts, both in pure and applied mathematics. However, many of the well-known results of classical linear algebra do not hold in this general setting. This is the case even if the underlying ring is

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a domain (but not a field). For a general introduction to matrix theory over commutative rings we refer to the textbook of Brown [4].

The purpose of this paper is to provide a better understanding of null ideals of square matrices over residue class rings of principal ideal domains.

Definition 1.1. Let S be a commutative ring, $A \in \mathrm{M}_n(S)$ an $n \times n$-square matrix A over S. The *null ideal* $\mathsf{N}^S(A)$ of A (over S) is the set of all polynomials which annihilate A, that is,

$$\mathsf{N}^S(A) = \{\, f \in S[X] \mid f(A) = 0 \,\}.$$

We often write $\mathsf{N}(A)$ instead of $\mathsf{N}^S(A)$ if the underlying ring is clear from the context.

In case S is a field, it is well-known that the null ideal of A is generated by a uniquely determined monic polynomial, the so-called *minimal polynomial* μ_A of A. Further, it is known that if S is a domain, then the null ideal of every square matrix is principal (generated by μ_A) if and only if S is integrally closed (Brown [5], Frisch [9]). However, little is known about the null ideal of a matrix with entries in a commutative ring. The well-known Cayley–Hamilton Theorem states that every square matrix over a commutative ring satisfies its own characteristic equation (cf. [12, Theorem XIV.3.1]). Therefore there always exists a monic polynomial in $S[X]$ of minimal degree which annihilates the matrix.

Definition 1.2. Let $A \in \mathrm{M}_n(S)$ be a square matrix over a commutative ring S. If $f \in S[X]$ is a monic polynomial with $f(A) = 0$ and there exists no monic polynomial in $S[X]$ of smaller degree with this property, then we call f a *minimal polynomial* of A over S.

Note that, in case S is a field, the definition above is consistent with the classical definition of the (uniquely determined) minimal polynomial of a square matrix. However in general, if S is not a field, a minimal polynomial of a matrix over S is not uniquely determined, although its degree is. It is known that if S is a domain, then the null ideal of A is principal if and only if A has a uniquely determined minimal polynomial over S, which is in turn equivalent to the (uniquely determined) minimal polynomial μ_A of A over the quotient field of S being in $S[X]$.

Brown discusses conditions for the null ideal to be principal over a general commutative ring R (with identity). In [7], he gives sufficient conditions on certain $R[X]$-submodules of the null ideal for the null ideal to be principal. There is also earlier work of Brown ([5,6]) investigating the relationship of the null ideals of certain pairs of square matrices over a commutative ring (which he refers to as spanning rank partners).

A better understanding of null ideals of matrices over residue class rings of domains has applications in the theory of integer-valued polynomials on matrix rings. Let D be a domain with quotient field K, and let $A \in \mathrm{M}_n(D)$. For a polynomial $f \in K[X]$, the image $f(A)$ of A under f is a matrix with entries in K. There are two immediate questions in