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Isotonic regression and isotonic projection

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ABSTRACT

The note describes the cones in the Euclidean space admitting isotonic metric projection with respect to the coordinate-wise ordering. As a consequence it is shown that the metric projection onto the isotonic regression cone (the cone defined by the general isotonic regression problem) admits a projection which is isotonic with respect to the coordinate-wise ordering.

1. Introduction

The isotonic regression problem [1,2,6,7,11,13,16] and its solution is intimately related to the metric projection into a cone of the Euclidean vector space. In fact the isotonic regression problem is a special quadratic optimization problem. It is desirable to relate the metric projection onto a closed convex set to some order theoretic properties of the

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projection itself, which can facilitate the solution of some problems. When the underlying set is a convex cone, then the most natural is to consider the order relation defined by the cone itself. This approach gives rise to the notion of the isotonic projection cone, which by definition is a cone with the metric projection onto it isotonic with respect to the order relation endowed by the cone itself. As we shall see, the two notions of isotonicity, the first related to the regression problem and the second to the metric projection, are at the first sight rather different. The fact that the two notions are in fact intimately related (this relation constitutes the subject of this note) is somewhat accidental and it derives from semantical reasons.

The relation of the two notions is observed and taken advantage in the paper [3]. There was exploited the fact that the totally ordered isotonic regression cone is an isotonic projection cone too.

The problem occurs as a particular case of the following more general question: *What does a closed convex set in the Euclidean space which admits a metric projection isotonic with respect to some vectorial ordering on the space look like?*

It turns out that the problem is strongly related to some lattice-like operations defined on the space, and in particular to the Euclidean vector lattice theory. (See [8].) When the ordering is the coordinate-wise one, the problem goes back in the literature to [4,9,10,14,15]. However, we shall ignore these connections in order to simplify the exposition. Thus, the present note, besides proving some new results, has the role to bring together some previous results and to present them in a simple unified form.

2. Preliminaries

Denote by $\mathbb{R}^m$ the m-dimensional Euclidean space endowed with the scalar product $\langle \cdot, \cdot \rangle : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$, and the Euclidean norm $\|.\|$ and topology this scalar product defines.

Throughout this note we shall use some standard terms and results from convex geometry (see e.g. [12]).

Let K be a *convex cone* in $\mathbb{R}^m$, i.e., a nonempty set with (i) $K + K \subset K$ and (ii) $tK \subset K$, $\forall\, t \in \mathbb{R}_+ = [0, +\infty)$. The convex cone K is called *pointed*, if $K \cap (-K) = \{0\}$. The cone K is *generating* if $K - K = \mathbb{R}^m$. K is generating if and only if $\operatorname{int} K \neq \emptyset$.

A closed, pointed generating convex cone is called *proper*.

For any $x, y \in \mathbb{R}^m$, by the equivalence $x \leq_K y \Leftrightarrow y - x \in K$, the convex cone K induces an *order relation* $\leq_K$ in $\mathbb{R}^m$, that is, a binary relation, which is reflexive and transitive. This order relation is *translation invariant* in the sense that $x \leq_K y$ implies $x + z \leq_K y + z$ for all $z \in \mathbb{R}^m$, and *scale invariant* in the sense that $x \leq_K y$ implies $tx \leq_K ty$ for any $t \in \mathbb{R}_+$. Conversely, if $\preceq$ is a translation invariant and scale invariant order relation on $\mathbb{R}^m$, then $\preceq = \leq_K$ with $K = \{x \in \mathbb{R}^m : 0 \preceq x\}$ a convex cone. If K is pointed, then $\leq_K$ is *antisymmetric* too, that is $x \leq_K y$ and $y \leq_K x$ imply that $x = y$. Conversely, if the translation invariant and scale invariant order relation $\preceq$ on $\mathbb{R}^m$ is also antisymmetric, then the convex cone $K = \{x \in \mathbb{R}^m : 0 \preceq x\}$ is also pointed. (In fact it