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Inequalities on the spectral abscissa for matrices arising in a stage-structured population model

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The spectral abscissa of a convex combination of two matrices of special form is compared with the convex combination of their individual spectral bounds. The matrices are essentially nonnegative and both the special form and the form of the comparison arise from a linear stage-structured population model. By using $\mathcal{M}$-matrix theory, the problem is converted to a comparison of the spectral radii of two related matrices. Theoretical results are derived for special cases of an arbitrary number of stage classes, and in particular confirm all previous numerical observations for the model with two stage classes.

1. Introduction

A recent paper [5] developed a linear (density independent) model for a structured ecological population with two stage classes, juveniles and adults, that lives in an environment that varies periodically in time between good and bad states in the case of either very rapid or very slow environmental variation. The dynamics of this model, and

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its extension to more stage classes, lead to interesting conjectures about the spectral abscissa for linear combinations of matrices that we both extend and prove here. Determining growth rates for this particular structured model is a special case of more general questions of the role of environmental variability in determining the growth rate and persistence of structured populations [7]. As noted in [7], this problem is very difficult in general and results are often counter-intuitive, so even partial clear-cut results are important. From an empirical point of view, the importance of resource pulses and more general time varying resources for persistence and dynamics of ecological populations has been recently reviewed and emphasized [8]. Since the general problem of time varying resources is so complex, the simpler case of either very rapid or very slow variation of resources is a first logical model to study and is the subject of the current paper.

In the continuous time density independent population dynamic model, let y_1 denote the population numbers in the first stage class into which all individuals are born with the numbers in subsequent stage classes denoted by y_i with i ranging from 2 to n. Then the population numbers are described by the vector

$$Y = (y_1, \dots, y_n)^T. \tag{1.1}$$

Assume that in the good environment, the per capita mortality rate of an individual in stage i is given by μ_i, the rate of maturation from stage i to $i+1$ is given by γ_i, and the per capita fecundity (rate of production of new individuals) of an individual in stage i is m_i with $\mu_i, m_i > 0$ for $i = 1, \dots, n$, and $\gamma_i > 0$ for $i = 1, \dots, n-1$. In the bad environment, the parameters are the same except that there is no reproduction, so all $m_i = 0$.

The population dynamics can be described using two real matrices A and B as follows. The $n \times n$ matrix A is defined as

$$A = \begin{bmatrix} -\mu_1 - \gamma_1 + m_1 & m_2 & \cdots & \cdots & m_{n-1} & m_n \\ \gamma_1 & -\mu_2 - \gamma_2 & 0 & \cdots & 0 & 0 \\ 0 & \gamma_2 & \ddots & & \vdots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 & \vdots \\ \vdots & \vdots & & \ddots & -\mu_{n-1} - \gamma_{n-1} & 0 \\ 0 & 0 & & & \gamma_{n-1} & -\mu_n \end{bmatrix}.$$

Let $B = A - M$, where

$$M = (1, 0, \dots, 0)^T (m_1, \dots, m_n). \tag{1.2}$$

The model assumes that the environment varies in time, with the dynamics in the good environment given by

$$dY/dt = AY$$