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Accessible proof of standard monomial basis for coordinatization of Schubert sets of flags ☆

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A B S T R A C T

The main results of this paper are accessible with only basic linear algebra. Given an increasing sequence of dimensions, a flag in a vector space is an increasing sequence of subspaces with those dimensions. The set of all such flags (the flag manifold) can be projectively coordinatized using products of minors of a matrix. These products are indexed by tableaux on a Young diagram. A basis of "standard monomials" for the vector space generated by such projective coordinates over the entire flag manifold has long been known. A Schubert variety is a subset of flags specified by a permutation. Lakshmibai, Musili, and Seshadri gave a standard monomial basis for the smaller vector space generated by the projective coordinates restricted to a Schubert variety. Reiner and Shimozono made this theory more explicit by giving a straightening algorithm for the products of the minors in terms of the right key of a Young tableau. Since then, Willis introduced scanning tableaux as a more direct way to obtain right keys. This paper uses scanning tableaux to give more-direct proofs of the spanning and the linear independence of the standard monomials. In the appendix it is noted that this basis is a weight basis for the dual of a Demazure module for a Borel subgroup of GL_n. This paper contains a complete proof that the characters of these modules (the key polynomials) can be

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expressed as the sums of the weights for the tableaux used to index the standard monomial bases.

1. Introduction

The main results of this paper are accessible to anyone who knows basic linear algebra: the Laplace expansion of a determinant is the most advanced linear algebra technique used. Otherwise, the most sophisticated fact needed is that the application of a multivariate polynomial may be moved inside a limit. Readers may replace our field $\mathbb{C}$ with any field of characteristic zero, such as $\mathbb{R}$.

Let $n \geq 2$ and $1 \leq k \leq n-1$. Fix $0 < q_1 < q_2 < \cdots < q_k < n$ and let Q denote the set $\{q_1, \ldots, q_k\}$. A Q-flag of $\mathbb{C}^n$ is a sequence of subspaces $V_1 \subset V_2 \subset \cdots \subset V_k \subset \mathbb{C}^n$ such that $\dim(V_j) = q_j$ for $1 \leq j \leq k$. The set $\mathcal{F}\ell_Q$ of Q-flags has long been studied by geometers. It is known as a *flag manifold (for GL_n)*. Given a fixed sequence of integers $\zeta_1 \geq \zeta_2 \geq \cdots \geq \zeta_m$ with $\zeta_i \in Q$ for $1 \leq i \leq m$, one can form projective coordinates for $\mathcal{F}\ell_Q$ as follows: First, any flag can be represented with a sequence of n column vectors of length n. The juxtaposition of these vectors forms an $n \times n$ matrix f. For each $1 \leq i \leq m$, form a left-initial $\zeta_i \times \zeta_i$ minor of f by selecting ζ_i of its n rows. We refer to a product of such minors as a "monomial" for the given ζ_j's. Let N be the number of such possible monomials. One can inefficiently coordinatize $\mathcal{F}\ell_Q$ in $\mathbb{P}(\mathbb{C}^N)$ by evaluating all of these monomials over the flag manifold. The sequence $\zeta_1, \ldots, \zeta_m$ can be viewed as the lengths of the columns of a Young diagram λ. Hodge and Pedoe [6] used a basis theorem of Young [18] to index an efficient subset of these coordinates with the semistandard Young tableaux on the diagram λ. This subset is a basis of "standard" monomials for the vector space spanned by all monomials over the flag manifold. One can group flags into subsets known as *Schubert varieties* using a form of Gaussian elimination on their matrix representatives; these can be indexed by n-permutations. For a given Schubert variety, the coordinatization by the set of monomials indexed by semistandard tableaux is inefficient. Utilizing recent developments in tableau combinatorics, this paper gives a new derivation of a basis of standard monomials for the vector space generated by all monomials restricted to a Schubert variety.

The most famous flag manifolds are the sets of d-dimensional subspaces of $\mathbb{C}^n$. These are the cases $k := 1$ and $q_1 := d$ above and are known as the Grassmannians. Here the basis result for Schubert varieties may be readily deduced once it is known for the entire Grassmannian. The next-most studied flag manifold is the "complete" flag manifold, which is the case $k := n-1$ above.

It was not until the late 1970s that Lakshmibai, Musili, and Seshadri first gave [9] a standard monomial basis for any Schubert variety of a general flag manifold (for GL_n). Their solution used sophisticated geometric methods and was expressed in the