

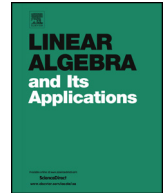


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Accessible proof of standard monomial basis for coordinatization of Schubert sets of flags[☆]



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ABSTRACT

The main results of this paper are accessible with only basic linear algebra. Given an increasing sequence of dimensions, a flag in a vector space is an increasing sequence of subspaces with those dimensions. The set of all such flags (the flag manifold) can be projectively coordinatized using products of minors of a matrix. These products are indexed by tableaux on a Young diagram. A basis of “standard monomials” for the vector space generated by such projective coordinates over the entire flag manifold has long been known. A Schubert variety is a subset of flags specified by a permutation. Lakshmibai, Musili, and Seshadri gave a standard monomial basis for the smaller vector space generated by the projective coordinates restricted to a Schubert variety. Reiner and Shimozono made this theory more explicit by giving a straightening algorithm for the products of the minors in terms of the right key of a Young tableau. Since then, Willis introduced scanning tableaux as a more direct way to obtain right keys. This paper uses scanning tableaux to give more-direct proofs of the spanning and the linear independence of the standard monomials. In the appendix it is noted that this basis is a weight basis for the dual of a Demazure module for a Borel subgroup of GL_n . This paper contains a complete proof that the characters of these modules (the key polynomials) can be

[☆] To be contained in the author’s doctoral thesis written under the supervision of Robert A. Proctor.

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expressed as the sums of the weights for the tableaux used to index the standard monomial bases.

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1. Introduction

The main results of this paper are accessible to anyone who knows basic linear algebra: the Laplace expansion of a determinant is the most advanced linear algebra technique used. Otherwise, the most sophisticated fact needed is that the application of a multivariate polynomial may be moved inside a limit. Readers may replace our field \mathbb{C} with any field of characteristic zero, such as \mathbb{R} .

Let $n \geq 2$ and $1 \leq k \leq n - 1$. Fix $0 < q_1 < q_2 < \cdots < q_k < n$ and let Q denote the set $\{q_1, \dots, q_k\}$. A Q -flag of \mathbb{C}^n is a sequence of subspaces $V_1 \subset V_2 \subset \cdots \subset V_k \subset \mathbb{C}^n$ such that $\dim(V_j) = q_j$ for $1 \leq j \leq k$. The set \mathcal{Fl}_Q of Q -flags has long been studied by geometers. It is known as a *flag manifold (for GL_n)*. Given a fixed sequence of integers $\zeta_1 \geq \zeta_2 \geq \cdots \geq \zeta_m$ with $\zeta_i \in Q$ for $1 \leq i \leq m$, one can form projective coordinates for \mathcal{Fl}_Q as follows: First, any flag can be represented with a sequence of n column vectors of length n . The juxtaposition of these vectors forms an $n \times n$ matrix f . For each $1 \leq i \leq m$, form a left-initial $\zeta_i \times \zeta_i$ minor of f by selecting ζ_i of its n rows. We refer to a product of such minors as a “monomial” for the given ζ_j ’s. Let N be the number of such possible monomials. One can inefficiently coordinatize \mathcal{Fl}_Q in $\mathbb{P}(\mathbb{C}^N)$ by evaluating all of these monomials over the flag manifold. The sequence ζ_1, \dots, ζ_m can be viewed as the lengths of the columns of a Young diagram λ . Hodge and Pedoe [6] used a basis theorem of Young [18] to index an efficient subset of these coordinates with the semistandard Young tableaux on the diagram λ . This subset is a basis of “standard” monomials for the vector space spanned by all monomials over the flag manifold. One can group flags into subsets known as *Schubert varieties* using a form of Gaussian elimination on their matrix representatives; these can be indexed by n -permutations. For a given Schubert variety, the coordinatization by the set of monomials indexed by semistandard tableaux is inefficient. Utilizing recent developments in tableau combinatorics, this paper gives a new derivation of a basis of standard monomials for the vector space generated by all monomials restricted to a Schubert variety.

The most famous flag manifolds are the sets of d -dimensional subspaces of \mathbb{C}^n . These are the cases $k := 1$ and $q_1 := d$ above and are known as the Grassmannians. Here the basis result for Schubert varieties may be readily deduced once it is known for the entire Grassmannian. The next-most studied flag manifold is the “complete” flag manifold, which is the case $k := n - 1$ above.

It was not until the late 1970s that Lakshmibai, Musili, and Seshadri first gave [9] a standard monomial basis for any Schubert variety of a general flag manifold (for GL_n). Their solution used sophisticated geometric methods and was expressed in the

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