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# A characterization of von Neumann rings in terms of linear systems ☆

Noemí DeCastro-García [b,*], Miguel V. Carriegos [a], Ángel Luis Muñoz Castañeda [c]

[a] *Instituto de Ciencias Aplicadas a Ciberseguridad, Departamento de Matemáticas, Universidad de León, Spain*
[b] *Departamento de Matemáticas, Universidad de León, Spain*
[c] *Institut für Mathematik, Freie Universität Berlin, Germany*

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A new characterization of commutative von Neumann regular rings is given in terms of linear control systems having, locally, a Brunovsky Canonical Form. The problem of enumerating reachable systems over certain von Neumann regular ring is solved extending classical results over fields.



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* Corresponding author.
*E-mail addresses:* ncasg@unileon.es (N. DeCastro-García), miguel.carriegos@unileon.es (M.V. Carriegos), angel@math.fu-berlin.de (Á.L. Muñoz Castañeda).

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## 1. Introduction

Feedback equivalence of linear systems over commutative rings has been largely studied using commutative algebra (see [4], [11], [15] or [24]). On the other hand the class of von Neumann regular rings has been studied in linear control systems framework (see [12,19–21]) and thus it is an active research field.

It is well known that Brunovsky's Theorem states that every reachable linear system over a field $\mathbb{K}$ is feedback equivalent to a Brunovsky Canonical Form, see [5,17,22]. It has also been proven in [8] that the class of rings where every reachable system is Brunovsky is exactly the class of fields.

Feedback invariants of linear systems are extended to the general case of commutative rings, see [16]. These invariants are shown to be complete for locally Brunovsky systems, that is to say for linear systems having locally a Brunovsky Canonical Form.

In this paper we prove that, over a commutative von Neumann regular ring every reachable system is locally of Brunovsky type and that this property does characterize the class of commutative von Neumann regular rings.

Maybe the first example of commutative von Neumann regular ring is the ring of modular integers $\mathbb{Z}/m\mathbb{Z}$ where $m = p_1 \dots p_t$ is square free integer. Another interesting example of von Neumann regular ring by its applications is the class of Boolean rings, that is, a ring $(\mathbb{B}, +, \cdot)$ such that $b^2 = b$ for all $b \in \mathbb{B}$. The Boolean networks, for example, are quite useful in modeling and quantitative description of cell regulation. A new technique has been developed for analyzing and synthesizing Boolean (control) networks based on the conversion of the logical dynamics of a Boolean network into a standard discrete-time dynamics, see [13].

Since there are classification and enumeration results on locally Brunovsky linear systems, see [7], these results may be brought directly to reachable systems over a commutative von Neumann regular rings. Hence we perform an count/enumeration of all reachable linear systems over commutative von Neumann regular rings.

The paper is organized as follows: Section 2 is devoted to preliminary results dealing both with von Neumann regular rings and locally Brunovsky systems. Then in Section 3 we give our main characterization result. The computation of the count of feedback equivalence classes of reachable systems is performed in Section 4. Finally, we give our conclusions.

## 2. Preliminaries

Let $R$ be a commutative non-trivial ring ($1 \neq 0$). Let $X$ be a finitely generated $R$-module, $f : X \longrightarrow X$ a linear map, and $B \subseteq X$ a $R$-submodule; and consider the linear system $\Sigma = (X, f, B)$ over $R$.

### *2.1. Locally Brunovsky linear systems*

Invariants $Z_i$ associated to $\Sigma$ given by the modules