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Matroid invariants and counting graph homomorphisms

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ABSTRACT

The number of homomorphisms from a finite graph F to the complete graph K_n is the evaluation of the chromatic polynomial of F at n. Suitably scaled, this is the Tutte polynomial evaluation $T(F; 1-n, 0)$ and an invariant of the cycle matroid of F. De la Harpe and Jaeger [8] asked more generally when is it the case that a graph parameter obtained from counting homomorphisms from F to a fixed graph G depends only on the cycle matroid of F. They showed that this is true when G has a generously transitive automorphism group (examples include Cayley graphs on an abelian group, and Kneser graphs).

Using tools from multilinear algebra, we prove the converse statement, thus characterizing finite graphs G for which counting homomorphisms to G yields a matroid invariant. We also extend this result to finite weighted graphs G (where to count homomorphisms from F to G includes such problems as counting nowhere-zero flows of F and evaluating the partition function of an interaction model on F).

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1. Introduction

1.1. Graph invariants and matroid invariants

A graph F in this paper will be finite and may have multiple edges and loops, i.e., by a graph we mean a finite multigraph. The set of vertices of F is denoted by $V(F)$ and the set of edges by $E(F)$. (Parallel edges appear with multiplicity.) The *cycle matroid* of a graph F is the matroid whose circuits are the edge sets of circuits in F. A loop of F is a circuit of size 1, and two parallel edges form a circuit of size 2. A bridge in F is a coloop in its cycle matroid. A matroid that is the cycle matroid of some graph is a *graphic matroid.* A standard reference for matroid theory is [12].

A *graph invariant* is a function defined on graphs with the property that it takes the same value on isomorphic graphs. A graph invariant taking values in a field such as $\mathbb{R}$ is also called a *graph parameter*. A graph invariant taking values in a polynomial ring is a *graph polynomial*, important examples being the chromatic polynomial and the Tutte polynomial (for which see for example [2,3]). A *matroid invariant* is a function defined on matroids with the property that it takes the same value on isomorphic matroids.

Matroids were introduced by Whitney in 1935 as an abstraction of the notion of independence in linear algebra and graph theory (a subset of edges of a graph being independent if it contains no cycle). Matroid theory permits the transfer of notions defined for one type of combinatorial structure to another seemingly unrelated one. There is therefore great interest in an invariant defined for a particular combinatorial structure (such as a graph) to which there is associated a matroid (the cycle matroid of a graph), as the invariant might be extended to a larger class of matroids.

A graph invariant whose value on a graph F depends only on the underlying cycle matroid of F will be called a *cycle matroid invariant.* A cycle matroid invariant is the restriction of a matroid invariant to graphic matroids, and for this reason de la Harpe and Jaeger [8] use the term matroid invariant for what we call a cycle matroid invariant. The reason for our terminological difference is explained in Section 4 below.

The Tutte polynomial of a graph F is a cycle matroid invariant and the chromatic polynomial of F scaled by a factor dependent only on the number of connected components of F is also a cycle matroid invariant (for example trees of the same size share the same chromatic polynomial). The Tutte polynomial of F may be defined in terms of the rank and size of subgraphs of F, which are terms defined for any matroid, and this definition can be used to define the Tutte polynomial as a matroid invariant, whose restriction to graphic matroids is the Tutte polynomial for graphs. This is one reason why the Tutte polynomial has played such a central role in combinatorics: matroids encompass a great diversity of combinatorial structures and the richly developed theory

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